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ACADEMICIAN V. L. MAKAROV IS 75 !

In 2016 the well-known Ukrainian scientist in the field of numerical mathematics, Academician of National Academy of Science of Ukraine (NASU), Doctor of Physical and Mathematical Sciences, Professor Volodymyr Leonidovych Makarov turned 75.

In 1963 V. L. Makarov graduated from the Faculty Mechanics and Mathematics of Kyiv State University. In 1967 he received a Ph. D. in Physics and Mathematics from the Kyiv State University. In 1974 he received a degree of a Doctor of Science in Physics, Mathematics and Computer Sciences and became a Professor of Applied and Computational Mathematics. In the period between 1981 and 1998 V. L. Makarov was the head of the Department of Numerical Methods of Mathematical Physics at the Kyiv National University of Ukraine. In October 1998 he became the head of Department of Numerical Mathematics at the Institute of Mathematics (NASU). During a long time V. L. Makarov was also the head of the Department of Applied Mathematics at the National Aviation University (Kyiv).

Numerous achievements of modern numerical mathematics are connected with the name of Professor V. L. Makarov. He developed many algorithms for solving different problems in mathematical physics; many other ones were developed and used for practical calculations under his supervision and with his direct participation. V. L. Makarov has also carried out a wide range of theoretical investigations in numerical mathematics. His works span a vast majority of problems in mathematical modeling including numerical simulation. These works have opened new directions in the theory of difference schemes, in automatic design of complex radio engineering systems etc. Professor Makarov developed the base of common theory of polynomial interpolation of non-linear operators in abstract spaces and recently obtained new important results in constructive representation of the solution operators for differential equations with operator coefficients in Hilbert and Banach spaces. The latter ones allow the construction of efficient numerical algorithms without accuracy saturation or exponential convergent algorithms for solving partial differential equations, integral equations etc.

Professor Makarov published more than 370 papers, 13 monographs and 8 textbooks. Since 1963 until 1974 the main direction of V. L. Makarov's scientific activities was the theory of difference schemes. In this period, he was onethe first to introduce and study the new class of difference schemes – a so-called difference scheme with exact and explicit spectrum. Studying the mathematical apparatus of these schemes, special functions of discrete argument, V. L. Makarov achieved some important results in the theory of associated orthogonal polynomials. Difference schemes with exact spectrums are widely used in practice, especially when solving hyperbolic equations with non-smooth solutions.

V. L. Makarov made an important contribution to the theory of exact and truncated differences schemes, the base of which was established in 1959-1968 by academicians A. M. Tikhonov and O. A. Samarskiy. These scientists and their followers proved the existence and uniqueness theorems for exact differences scheme for vectorial systems of ordinary differential equations of the second order, of ordinary differential equations of the forth order, differential equations with degeneration on the boundary and in unbounded domains. Sufficient conditions for conservatism of differences scheme for the equations of gasdynamics were pointed out.

In 1979-1980 in their common works, V. L. Makrov and academician O. A. Samarskiy suggested a new direction in numerical mathematics, namely difference schemes which rate of convergence is adjusted to the smoothness of the solution of the primary differential problem. These investigations were continued by V. L. Makarov and his followers. They derived and studied differences scheme with adjusted convergencerate for quasi-linear problems of mathematical physics in Sobolev spaces. Now these models are widely used in mechanics, elasticity theory, theory of operating systems with distributed parameters etc.

Since 1975 V. L. Makarov engaged in active research on the development of theoretical base for automatic projection of complicated radio engineering systems. This research, under his supervision and with his direct participation, created the mathematical concept of systems of embedded models, methods of verification of mathematical models, the statistical approach to the problem of verification. Important attention was paid to the algorithmic realization of mathematical models, where the results by V. L. Makarov in the field of numerical methods were used.

During the last years V. L. Makarov laid the foundation of the general theory of the polynomial interpolation of the non-linear operators in abstract spaces. His work proves the necessary and sufficient existence and uniqueness conditions for polynomial interpolants in Hilbert and vector spaces and proposes procedures to construct these polynomials. Professor Makarov also obtained generalizations for the case of interpolation conditions containing Gato derivatives in all directions.

In the last decade V. L. Makarov have proposed and futher develops a very efficient so called FD-method, which shows especially good results for eigenvalue problems.

In 1990 Prof. Makarov, while working in an international team at the University of Leipzig, began a new line of research. Together with I. P. Gavrilyuk he studied differential equations with operator coefficients as meta-models of partial differential equations, their solution operators and various operator equations in Hilbert and Banach spaces. A series of results of fundamental importance were obtained by Professor V. L. Makarov in this. These results were the base for the new efficient parallel approximations without accuracy saturation or with an exponential convergence rate to solutions of various partial differential equations. The exponentially convergent methods for various mathematical and applied problems remain to be the focus of Professor Makarovs research activities of the last decade since they are the basis foralgorithms of optimal complexity. A part of results on this was published in the Birkhauser Series "Frontiers in Mathematics" (in co-authorship with I. Gavrilyuk and V. Vasylyk).

An important field of Professor Makarovs scientific activities is mathematical modeling of sloshing of fluids in moving containers with various marine applications. These phenomena are described by complex systems of nonlinear partial differential equations in domains with moving boundary. This investigations of Professor Makarov were supported by the German Research Council (DFG) and by the German Academic Council (DAAD). Professor V. L. Makarov has been teaching for 35 years in the Taras Shevchenko National University of Kyiv. He created a school of numerical mathematics which includes 48 candidates (PhD) and 15 doctors (DS) of physical-mathematical sciences that have prepared their theses under his supervision. Results published by V. L. Makarov are widely known in the scientific world and make an important contribution to mathematics. Long before the end of the Soviet Union V. L. Makarov has prepared the first teaching complex of books on numerical methods in Ukrainian (in co-authorship) including two theoretical parts, a practical part with algorithms and programs as well as two books with a collection of exercises. At that time such a complex was a novelty in teaching of numerical mathematics and not only in Ukraine. A creative and fruitful relationship connects V. L. Makarov with many other scientists including the famous mathematical schools of academicians A. M. Tikhonov, O. A. Samarskiy, and Kyiv and Leipzig schools of numerical and applied mathematics.

Under the guidance of Academician Makarov, a seminar on numerical mathematics takes place. V. L. Makarov is the editor ofthefollowing journals: "Differential equations", "Ukrainian Mathematical Journal", "Nonlinear Oscillations", CMAM, AMI, and a deputy editor-in-chief of the Journal of Numerical and Applied Mathematics. Besides, he repeatedly belonged to specialized boards of the doctoral and Ph. D. thesis defends. Academician Makarov was invited speaker at a number of international conferences and schools of applied mathematics. He is a member of the American Mathematical Society.

For the gained success in his work V. L. Makarov was awarded the order of the Labor Red Flag (1984), M. M. Krylov's Prize from NASU (2007), M. M. Bogolubov's Prize from NASU (2012) and State Prize of Ukraine in Science and Technology in 2012. In 2000 V. L. Makarov was elected as a Corresponding Member of the NASU and in 2009 as an Academician of the NASU.

Volodymyr Leonidovych Makarov is full of new scientific ideas and concepts. His active work promotes development of numerical mathematics in Ukraine and recognition of the achievements of Ukrainian mathematicians by the international scientific society.

We cordially congratulate the celebrator of a jubilee and wish Volodymyr Leonidovych creative successes and scientific longevity.

R. Chapko, I. Gavrilyuk, O. Khimich, V. Khlobystov, V. Korolyuk, M. Kutniv, I. Lukovskyy, P. Matus, A. Samoilenko, G. Shynkarenko, S. Solodkyy, O. Timokha, V. Vasylyk, M. Voitovich.

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ON THE NON-LINEAR INTEGRAL EQUATION APPROACHES FOR THE BOUNDARY RECONSTRUCTION IN DOUBLE-CONNECTED PLANAR DOMAINS

R. S. Chapko, O. M. Ivanyshyn Yaman, T. S. Kanafotskyi

Резюме. Розглядається задача реконструкції внутрішньої кривої за заданими даними Коші гармонійної функції на зовнішній кривій плоскої області. За допомогою функції Гріна і теорії потенціалу нелінійна обернена задача редукована до системи нелінійних граничних інтегральних рівнянь. Розроблено три ітераційні алгоритми для її чисельного розв'язування. Знайдено похідні Фреше відповідних операторів і показано єдиність розв'язку лінеаризованих систем. Повна дискретизація здійснена методом тригонометричних квадратур. Через некоректність вихідної задачі до отриманих систем лінійних рівнянь застосовано регуляризацію Тіхонова. Чисельні результати показують, що пропоновані методи дають достатньо добру точність реконструкції при економних обчислювальних затратах.

Abstract. We consider the reconstruction of an interior curve from the given Cauchy data of a harmonic function on the exterior boundary of the planar domain. With the help of Green's function and potential theory the non-linear boundary reconstruction problem is reduced to the system of non-linear boundary integral equations. The three iterative algorithms are developed for its numerical solution. We find the Fréchet derivatives for the corresponding operators and show unique solviability of the linearized systems. Full discretization of the systems is realized by a trigonometric quadrature method. Due to the inherited ill-possedness in the obtained system of linear equations we apply the Tikhonov regularization.

The numerical results show that the proposed methods give a good accuracy of reconstructions with an economical computational cost.

1. Introduction

The mathematical modeling of electrostatic or thermal imaging methods in nondestructive testing and evaluation leads to inverse boundary value problems for the Laplace equation. In principle, in these applications an unknown inclusion within a conducting host medium with a constant conductivity is resolved from the overdetermined Cauchy data on the accessible part of the boundary of the medium.

The idea to reduce the problem of the boundary reconstruction to the system of non-linear equations and to employ a regularized iterative procedure was firstly suggested in [11]. This approach was successfully extended in [1, 3, 6, 11,

Key words. Double connected domains; boundary reconstruction; Green's function; single layer potential; boundary integral equations; trigonometric quadrature method; Tikhonov regularization.

12] for the case of the Laplace equation and in [4,5,7–9,13–15] for the Helmholtz equation.

As an alternative to the reciprocity gap approach based on Green's integral theorem we propose iterative solution methods based on the Green's function. Although the proposed methods are restricted to the class of domains for which the Green's function can be easily found the methods have several advantages over the reciprocity gap approach. In particular, the corresponding single layer potential is bounded at infinity and hence its modification is not needed. Moreover, for the complicated boundary conditions such as generalized impedance the proposed methods will be easier to adopt.

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ and Λ such that Γ is contained in the interior of Λ.

The corresponding direct problem is: Given a function f on Λ consider the Dirichlet problem for $u \in C^2(D) \cap C(\bar{D})$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in } D \tag{1}$$

and the boundary conditions

$$u = 0 \quad \text{on } \Gamma, \tag{2}$$

$$u = f \quad \text{on } \Lambda. \tag{3}$$

The inverse problem we are concerned with is: Given the Dirichlet data f on Λ with $f \neq 0$ and the Neumann data

$$g := \frac{\partial u}{\partial \nu} \quad \text{on } \Lambda, \tag{4}$$

determine the shape of the interior boundary Γ. Here, and in the sequel, by ν we denote the outward unit normal to Γ or to Λ. We tacitly assume that f has enough smoothness, for example $f \in C^{1,\alpha}(\Lambda)$ for classical solutions or $f \in H^{1/2}(\Lambda)$ for weak solutions, to ensure the existence of the normal derivative on Λ. As opposed to the forward boundary value problem, the inverse problem is nonlinear and ill-posed.

The issue of uniqueness, i.e., identifiability of the unknown curve Γ from the Cauchy data on Λ, is settled by the following theorem (see [10]).

Theorem 1. *Let Γ and $\widetilde{\Gamma}$ be two closed curves contained in the interior of Λ and denote by u and $\widetilde{u}$ the solutions to the Dirichlet problem* (1)–(3) *for the interior boundaries Γ and $\widetilde{\Gamma}$, respectively. Assume that $f \neq 0$ and*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \widetilde{u}}{\partial \nu}$$

on an open subset of Λ. Then $\Gamma = \widetilde{\Gamma}$.

The plan of the paper is as follows. In Section 2 we reduce the inverse boundary value problem (1)–(4) to two boundary integral equations using Green's function. Section 3 contains three iterative schemes for the numerical solution of the non-linear integral equations. We show the injectivity of the corresponding linearized operators. The practical realization of suggested algorithms is discussed in Section 4. Section 5 concludes the paper with some numerical examples illustrating the feasibility of the non-linear integral equation method for approximate solution of the inverse boundary value problem.

2. Reduction to boundary integral equations

To this end, we denote the interior of Λ by B. Then, by G we denote the Green's function for B, that is, G is defined for all $x \neq y$ in $\overline{B}$ and of the form

$$G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \tilde{G}(x,y),$$

where, for a fixed $y \in B$, the function $\tilde{G}$ is harmonic in B with respect to x such that $G(\cdot\,,y) = 0$ on Λ. We note that for Λ a circle of radius R centered at the origin $\tilde{G}$ is explicitly given by

$$\tilde{G}(x,y) = \frac{1}{4\pi} \ln \frac{R^4 + |x|^2|y|^2 - 2R^2\, x \cdot y}{R^2} \; .$$

The solution w to the Dirichlet problem in B with boundary values $w = f$ on Λ can be represented in the form

$$w(x) = -\int_{\Lambda} \frac{\partial G(x,y)}{\partial \nu(y)} \, f(y)\, ds(y), \quad x \in B. \tag{5}$$

In the case of Λ a circle the representation (5) reduces to the Poisson integral. In a more abstract sense, we may interpret (5) as solution operator that maps the boundary value f into the solution w of the Dirichlet problem in B. Seeking the unique solution of (1)–(3) in the form

$$u(x) = \int_{\Gamma} G(x,y)\varphi(y)\, ds(y) + w(x), \quad x \in D, \tag{6}$$

now leads to the integral equation of the first kind

$$\int_{\Gamma} G(x,y)\varphi(y)\, ds(y) = -w(x), \quad x \in \Gamma, \tag{7}$$

for the unknown density φ. We name the integral equation (7) as a field equation. The given flux g on Λ leads to the integral equation

$$\int_{\Gamma} \varphi(y) \frac{\partial G(x,y)}{\partial \nu(x)} ds(y) = g(x) - \frac{\partial w}{\partial \nu}(x), \quad x \in \Lambda, \tag{8}$$

which is named a data equation.

Let introduce the single-layer potential

$$(S\varphi)(x) := \int_{\Gamma} G(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma, \tag{9}$$

and the operator

$$(A\varphi)(x) := \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(x)} \, \varphi(y)\, ds(y), \quad x \in \Lambda, \tag{10}$$

for the normal derivative of the single-layer potential on Λ.

Theorem 2. *The inverse boundary value problem* (1)–(4) *is equivalent to the system of integral equations*

$$S\varphi = -w \quad \textit{on} \quad \Gamma, \tag{11}$$

$$A\varphi = g - \frac{\partial w}{\partial \nu} \quad \textit{on} \quad \Lambda. \tag{12}$$

Proof. Analogously to [11]. □

Theorem 3. *The operator* $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bijective and has bounded inverse. The operator* $A : L^2(\Gamma) \to L^2(\Lambda)$ *is injective and has dense range.*

Proof. The bijectivity of S is the classical result and can be found in [10]. The injectivity of A is proved in [2]. □

To describe the algorithms conveniently a parametrization of boundary curves is required. Let

$$\lambda(s) = \{(x_1(s), x_2(s)) : s \in [0, 2\pi]\}$$

is the parametrization for the exterior curve Λ. For simplicity we consider only starlike interior curves, i.e., we choose a parametrization in polar coordinates of the form

$$\gamma_r(s) = \{r(s)c(s) : s \in [0, 2\pi]\}, \tag{13}$$

where $c(s) = (\cos s, \sin s)$ and $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin. However, we wish to emphasize that the concepts described below, in principle, are not confined to starlike boundaries only. We introduce the parametrized density as $\varphi(t) := \varphi(\gamma_r(t))$ or $\phi(t) := \varphi(\gamma_r(t))|\gamma_r'(t)|$. We indicate the dependence on r by denoting the curve with parametrization (13) by Γ_r. The corresponding operators defined through (9) and (10) for $\Gamma = \Gamma_r$ are given by

$$(S_r\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)G(\gamma_r(t), \gamma_r(\tau))d\tau,$$

$$(\tilde{S}_r\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)G(\gamma_r(t), \gamma_r(\tau))|\gamma_r'(\tau)|d\tau,$$

$$(A_r\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)\frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau))d\tau$$

and

$$(\tilde{A}_r\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau))|\gamma_r'(\tau)|d\tau.$$

3. Iterative schemes

Operators S_r, A_r and $\tilde{A}_r$ have the following Freéchet derivatives with respect to the radial function r

$$(S'[r, \phi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)[q(\tau)L_r^{(1)}(t, \tau) + q(t)L_r^{(2)}(t, \tau)]d\tau,$$

$$(A'[r, \phi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)q(\tau)H_r^{(1)}(t, \tau)d\tau.$$

and

$$(\tilde{A}'[r,\varphi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\Big[q(\tau)H_r^{(1)}(t,\tau)|\gamma_r'(\tau)| + \\ + \frac{r(\tau)q(\tau)+r'(\tau)q'(\tau)}{|\gamma_r'(t)|}H_r^{(2)}(t,\tau)\Big]d\tau. \tag{14}$$

Here we introduced the kernels

$$L_r^{(1)}(t,\tau) := -\frac{r(\tau)-r(t)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(\tau)} \tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(\tau),$$

$$L_r^{(2)}(t,\tau) := -\frac{r(t)-r(\tau)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(t)} \tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(t),$$

$$H_r^{(1)}(t,\tau) := \operatorname{grad}_{\gamma_r(\tau)} \frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}\cdot c(\tau)$$

and

$$H_r^{(2)}(t,\tau) := \frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}.$$

Note that

$$\lim_{\tau\to t}(q(\tau)L_r^{(1)}(t,\tau)+q(t)L_r^{(2)}(t,\tau)) = \frac{r(t)q(t)+r'(t)q'(t)}{|\gamma_r'(t)|^2} + \\ + 2q(t)\operatorname{grad}_{\gamma_r(t)}\tilde{G}(\gamma_r(t),\gamma_r(t))\cdot c(t).$$

These representation were obtained by standard differentiation procedure in (9) and (10). Also we will need the Freéchet derivative for the function w

$$(w'[r]q)(t) = -\frac{1}{2\pi}\int_0^{2\pi} f(\tau)q(\tau)W_r(t,\tau)d\tau$$

with

$$W_r(t,\tau) := |\lambda'(\tau)|\operatorname{grad}_{\gamma_r(t)}\frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}\cdot c(t).$$

The linear operators $S'[r,\varphi]$ and $A'[r,\varphi]$ have the following properties.

Theorem 4. *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation* (11)*, i.e. $S_r\phi = -w$ on Γ_r. Assume that $q \in C^2[0,2\pi]$ and $\psi \in L^2[0,2\pi]$ solve the homogeneous system*

$$S_r\psi + S'[r,\phi]q + w'[r]q = 0, \tag{15}$$

$$A_r\psi + A'[r,\phi]q = 0. \tag{16}$$

Then $q = 0$ and $\psi = 0$.

Proof. As it is shown in [6], for sufficiently small q, the perturbed interior curve as given in polar coordinates by

$$\Gamma_{r+q} = \{(r(t)+q(t))c(t) : t\in[0,2\pi]\}$$

can be represented in the form

$$\Gamma_{r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t\in[0,2\pi]\}$$

in terms of the normal vector

$$\nu(t) = r'(t)(-\sin t, \cos t) - r(t)(\cos t, \sin t)$$

to the unperturbed curve Γ_r and a function $\widetilde{q}$. Now in the Fréchet derivatives S', A' and w' we may replace the perturbation vector $\zeta(t) = q(t)c(t)$ by $\widetilde{\zeta} = \widetilde{q}\,\nu$. We introduce the function

$$V(x) := \int_0^{2\pi} \psi(\tau) G(x, \gamma_r(\tau)) d\tau - \\ - \int_0^{2\pi} \mathrm{grad}_x\, G(x, \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

Then (16) implies that $\frac{\partial V}{\partial \nu} = 0$ on Λ. The function V satisfies the Laplace equation in the exterior of Λ, it decays at infinity, therefore by the uniqueness for the exterior Neumann problem we conclude that $V \equiv 0$ in the exterior of Λ. By analyticity we obtain $V \equiv 0$ in the exterior of Γ_r. Approaching Γ_r from the exterior by the jump relations we obtain

$$0 = \int_0^{2\pi} \psi(\tau) G(\gamma_r(t), \gamma_r(\tau) d\tau \\ - \int_0^{2\pi} \mathrm{grad}_{\gamma_r(t)}\, G(\gamma_r(t), \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t), \quad t \in [0, 2\pi].$$

Employing the above equality and recalling the definition (6) of u we can rewrite (15) as follows

$$\widetilde{\zeta} \cdot \mathrm{grad}\, u \circ \gamma_r = 0.$$

Due to the definition of u and the condition on φ we have $u = 0$ on Γ_r, which is equivalent to

$$\widetilde{\zeta} \cdot \nu \circ \gamma_r \left(\frac{\partial u}{\partial \nu}\right) \circ \gamma_r = 0.$$

Since by Holmgren's theorem $\frac{\partial u}{\partial \nu}$ cannot vanish on open subsets of Γ_r we obtain $\widetilde{\zeta} \cdot \nu \circ \gamma_r = \widetilde{q} = 0$ and hence $q = 0$. Analogously to [11] by continuity of a single-layer potential and the uniqueness of the interior Dirichlet problem we obtain $V = 0$ in $\mathbb{R}^2$ and therefore the density $\psi = 0$. □

Theorem 5. *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation* (12)*, i.e. $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ. Assume that $q \in C^2[0, 2\pi]$ solves the homogeneous equation*

$$S'[r, \phi]q + w'[r]q = 0. \tag{17}$$

Then $q = 0$.

Proof. Since ϕ is a solution to $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ it also satisfies $S_r\phi = -w$ on Γ_r. We represent the perturbed interior curve again as

$$\Gamma_{r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t \in [0, 2\pi]\}$$

and introduce the function

$$V(x) := \int_0^{2\pi} \phi(\tau) G(x, \gamma_r(\tau)) d\tau - \int_\Lambda \frac{\partial G(x, y)}{\partial \nu(y)}\, f(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

The function V is a solution to the interior Dirichlet boundary value problem with the homogeneous condition. In view of the unique solution we obtain $V \equiv 0$ in the interior of Γ_r and therefore $\left.\dfrac{\partial V}{\partial \nu}\right|_{\Gamma_r} = 0$, i.e.

$$\begin{aligned} 0 = {} & \widetilde{q}(t)\nu(t) \cdot \operatorname{grad}_{\gamma_r(t)} \int_0^{2\pi} G(\gamma_r(t), \gamma_r(\tau))\phi(\tau)\, d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t) \\ & -\widetilde{q}(t)\nu(t) \cdot \operatorname{grad}_{\gamma_r(t)} \int_\Lambda \frac{\partial G(\gamma_r(t), y)}{\partial \nu(y)} f(y)\, ds(y), \quad t \in [0, 2\pi]. \end{aligned}$$

From (17) we find

$$0 = -\frac{1}{2}\widetilde{q}(t)\phi(t) - \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t), \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\, \phi(\tau)\, d\tau, \quad t \in [0, 2\pi]. \tag{18}$$

We define a double layer potential

$$W(x) := -\int_0^{2\pi} \operatorname{grad}_x G(x, \gamma_r(\tau)) \cdot \nu(\tau)\widetilde{q}(\tau)\phi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

Since the function W is harmonic in the interior of Γ_r and satisfies the homogeneous Dirichlet boundary condition, (18), it implies $W \equiv 0$ in the interior of Γ_r. One can show, similarly to [10, Theorem 6.21], that the operator $-I + K$ is injective, where

$$(K\psi)(t) = \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t), \gamma_r(\tau)) \cdot \nu(\tau)\psi(\tau)\, d\tau, \quad t \in [0, 2\pi]$$

Hence from (18) we obtain

$$\widetilde{q}(t)\phi(t) = 0, \quad t \in [0, 2\pi]$$

By the jump relations for the function V we have

$$\frac{1}{|\gamma_r'|}\phi = \left.\frac{\partial V^-}{\partial \nu}\right|_{\Gamma_r} - \left.\frac{\partial V^+}{\partial \nu}\right|_{\Gamma_r} = -\left.\frac{\partial V^+}{\partial \nu}\right|_{\Gamma_r}.$$

Since by Holmgren's theorem $\frac{\partial V^+}{\partial \nu}$ cannot vanish on open subsets of Γ_r and $|\gamma_r| \neq 0$ we obtain $\widetilde{q} = 0$ and hence $q = 0$. □

Remark (about the Algorithm 2).

If the interior boundary is a circle, then exists a nontrivial solution $q = const$ to the homogeneous equation $A'[r, \varphi]q = 0$. Indeed, introducing the function

$$V(x) = -q \operatorname{grad}_x \int_0^{2\pi} G(x, \gamma_r(\tau)) \cdot \nu(\tau)\varphi(\tau) d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r$$

we obtain that V is a unique solution to the Neumann boundary value problem with the homogeneous condition in the exterior of Λ, and hence $V^+|_{\Gamma_r} = 0$. Since the null-space of the operator of the integral equation

$$\frac{1}{2}\varphi(t) - \operatorname{grad}_x \int_0^{2\pi} G(t, \gamma_r(\tau)) \cdot \nu(\tau)\varphi(\tau) d\tau = 0, \quad t \in [0, 2\pi]$$

is not empty, one can find $q \neq 0$ which solves $A'[r, \varphi]q = 0$.

In view of this remark we introduced the modified version $\widetilde{A}'[r,\varphi]$, (14), instead of the operator $A'[r,\varphi]$.

Now we describe three iterative algorithms for the numerical solution of (11)-(12).

Algorithm 1.

1. Choose some starting value r. Solve the well-posed integral equation

$$S_r\phi = -w_r. \tag{19}$$

2. For the given r and φ solve the system of linearized ill-posed integral equations

$$S_r\psi + S'[r,\phi]q + w'[r]q = -S_r\phi - w_r, \tag{20}$$

$$A_r\psi + A'[r,\phi]q = g - \frac{\partial w}{\partial \nu} - A_r\phi \tag{21}$$

with respect to functions ψ and q.

3. Calculate new approximations for the radial function $r = r + q$ and for the density $\phi = \phi + \psi$.

4. Repeat steps 2-3 until some stopping criterion is satisfied.

Algorithm 2.

1. Choose some starting value r.

2. Solve the well-posed integral equation

$$\tilde{S}_r\varphi = -w_r. \tag{22}$$

3. For the given r and φ solve the linearized ill-posed integral equation

$$\tilde{A}'[r,\varphi]q = g - \frac{\partial w}{\partial \nu} - \tilde{A}_r\varphi \tag{23}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.

5. Repeat steps 2-4 until some stopping criterion is satisfied.

Algorithm 3.

1. Choose some starting value r.

2. Solve the ill-posed integral equation

$$A_r\phi = g - \frac{\partial w}{\partial \nu}. \tag{24}$$

3. For given r and φ solve the linearized ill-posed integral equation

$$S'[r,\phi]q + w'[r]q = -S_r\phi - w_r, \tag{25}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.

5. Repeat steps 2-4 until some stopping criteria is satisfied. Note here that we need to use some regularization method in the case of ill-posed integral equations. According to properties of the corresponding integral operators an application of the Tikhonov regularization is justified for the algorithms $1, 3$.

4. Implementation

Algorithm 1.

Step1. On the first step of this algorithm we need to solve the well posed integral equation of the first kind (19) with a logarithmic singularity for a current approximation of r. Since all functions in this equation are 2π periodic we implement the trigonometric quadrature method. To do this we rewrite the equation (19) in the following equivalent form

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(\tau)\left[-\frac{1}{2}\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+K_r(t,\tau)\right]d\tau=-w_r(t),\quad t\in[0,2\pi],$$

where

$$K_r(t,\tau):=\frac{1}{2}\ln\frac{\frac{4}{e}\sin^2\frac{t-\tau}{2}}{|\gamma_r(t)-\gamma_r(\tau)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(\tau)),\ \ t\neq\tau$$

with the diagonal term

$$K_r(t,t)=\frac{1}{2}\ln\frac{1}{e|\gamma_r'(t)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(t)).$$

The following trigonometric quadratures with equidistant points $t_j=\frac{j\pi}{n}$, $j=0,\dots,2n-1$ are used

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau\approx\sum_{k=0}^{2n-1}R_k(t)\,f(t_k)\tag{26}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\,d\tau\approx\frac{1}{2n}\sum_{k=0}^{2n-1}f(t_k)\tag{27}$$

with explicit expressions for the weight functions given in [10]. It leads to the following system of linear equations with respect to $\phi_{ni}\approx\phi(t_i)$

$$\sum_{i=0}^{2n-1}\phi_{ni}[-\frac{1}{2}R_i(t_k)+\frac{1}{2n}K(t_k,t_i)]=-\tilde{w}_r(t_k),\ \ k=0,\dots,2n-1$$

with

$$\tilde{w}_r(t)=-\frac{1}{2n}\sum_{i=0}^{2n-1}f(t_i)H(t,t_i),$$

where

$$H(t,\tau):=\frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}|\lambda'(t)|.$$

The convergence and error analysis for this method can be found in [10].

Step2. We search the unknown corrections in the system (20)-(21) as

$$\psi_n=\sum_{i=0}^{2n-1}\psi_{ni}l_i^1,\quad q_m=\sum_{i=0}^{2m}q_{mi}l_i^2,$$

where l_i^1, $i = 0, \ldots, 2n-1$ are basic Lagrangian trigonometric polynomials and l_i^2, $i = 0, \ldots, 2m$ are known basic functions. The quadrature method applied to (20)-(21) give us the linear system

$$\sum_{i=0}^{2n-1} \psi_{ni} \mathcal{A}_{ki}^{(11)} + \sum_{i=0}^{2m} q_{mi} \mathcal{A}_{ki}^{(12)} = b_k^{(1)}, \quad k = 0, \ldots, 2n-1,$$

$$\sum_{i=0}^{2n-1} \psi_{ni} \mathcal{A}_{ki}^{(21)} + \sum_{i=0}^{2m} q_{mi} \mathcal{A}_{ki}^{(22)} = b_k^{(2)}, \quad k = 0, \ldots, 2n-1$$

with matrix coefficients

$$\mathcal{A}_{ki}^{(11)} = -\frac{1}{2} R_i(t_k) + \frac{1}{2n} K_r(t_k, t_i), \quad \mathcal{A}_{ki}^{(21)} = \frac{1}{2n} H_r^{(2)}(t_k, t_i),$$

$$\mathcal{A}_{ki}^{(12)} = \frac{1}{2n} \sum_{j=0}^{2n-1} \{\phi_{nj} [l_i^2(t_j) L_r^{(1)}(t_k, t_j) + l_i^2(t_k) L_r^{(2)}(t_k, t_j)] + l_i^2(t_j) f(t_i) W_r(t_k, t_j)\},$$

$$\mathcal{A}_{ki}^{(22)} = \frac{1}{2n} \sum_{j=0}^{2n-1} \phi_{nj} l_i^2(t_j) H_r^{(1)}(t_k, t_j)$$

and right hand side

$$b_k^{(1)} = \sum_{i=0}^{2n-1} \phi_{ni} [-\frac{1}{2} R_i(t_k) - \frac{1}{2n} K_r(t_k, t_i)] - \tilde{w}_r(t_k),$$

$$b_k^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k) - \frac{1}{2n} \sum_{i=0}^{2n-1} \phi_{ni} H_r^{(2)}(t_k, t_i).$$

Here $2n \geq 2m + 1$.
Thus the received ill-posed linear system is overdetermined and therefore we reduce it to the least-squares problem. The following cost functional needs to be minimized

$$F(\psi_{n0}, \ldots, \psi_{n,2n-1}, q_{m0}, \ldots, q_{m,2m}) =$$

$$= \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj} \mathcal{A}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj} \mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 +$$

$$\sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj} \mathcal{A}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj} \mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 +$$

$$\alpha \sum_{j=0}^{2n-1} \omega_{1j} \psi_{nj}^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2$$

with the regularization parameters $\alpha > 0$ and $\beta > 0$ and weight coefficients ω_{1j} and ω_{2j}. Clearly, the final linear system has the form

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(12)} = \mathbf{b}_i^{(1)}, \quad i = 0, \dots, 2n-1,$$

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(22)} = \mathbf{b}_i^{(2)}, \quad i = 0, \dots, 2m,$$

where

$$\mathbf{a}_{ij}^{(\ell p)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)}\mathcal{A}_{kj}^{(p1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)}\mathcal{A}_{kj}^{(p2)}$$

and

$$\mathbf{b}_i^{(\ell)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)} b_k^{(1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)} b_k^{(2)}.$$

Step 3. Now we can evaluate the new values for the radial function $r_m = r_m + q_m$ and for the density $\phi_n = \phi_n + \psi_n$.

The following stopping criterion can be used

$$\frac{\|q_m\|_{L^2[0,2\pi]\|}}{\|r_m\|_{L^2[0,2\pi]}} < \epsilon$$

with sufficiently small $\epsilon > 0$, or a discrepancy principle, as well.

Algorithm 2.

Step2. It is analogous to the *Step 1* from the Algorithm 1.

Step3. To find the correction q from (23) we make the discretization by the quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \dots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \; 2n \geq 2m+1.$$

The corresponding linear system has the form

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij} = \mathbf{b}_i, \quad i = 0, \dots, 2m$$

with

$$\mathbf{a}_{ij} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)}\mathcal{A}_{kj}^{(22)}, \quad \mathbf{b}_i = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)} b_k^{(2)}.$$

Algorithm 3.

Step2. The discretization in (24) and ill-posednes of the received linear system lead to the minimization of the following Tikhonov functional

$$F(\psi_{n0}, \dots, \psi_{n,2n-1}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj}\mathcal{A}_{ij}^{(21)} - \tilde{b}_i^{(2)} \right|^2 + \alpha \sum_{j=0}^{2n-1} \omega_{1j}\psi_{nj}^2$$

with

$$\tilde{b}_i^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k).$$

which is equivalent to solving the linear system

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(1)} = \mathbf{b}_i^{(1)}, \quad i = 0, \ldots, 2n-1,$$

where

$$\mathbf{a}_{ij}^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\mathcal{A}_{kj}^{(21)}, \quad \mathbf{b}_i^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\tilde{b}_k^{(2)}.$$

Step3. To find the correction q from (25) we make the discretization by quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \ldots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \; 2n \geq 2m+1.$$

Thus the corresponding linear system has the form

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(2)} = \mathbf{b}_i^{(2)}, \quad i = 0, \ldots, 2m,$$

where

$$\mathbf{a}_{ij}^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)}\mathcal{A}_{kj}^{(12)}, \quad \mathbf{b}_i^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)} b_k^{(1)}.$$

5. Numerical examples

We demonstrate the feasibility of the proposed methods for the inverse problem (1)-(4) with the following boundaries $\lambda(t) = \{Rc(t), t \in [0, 2\pi]\}$ with $R = 2$, and

$$\gamma_r(t) = \left\{ \sqrt{\cos^2 t + 0.25\sin^2 t}\, c(t), t \in [0, 2\pi] \right\}.$$

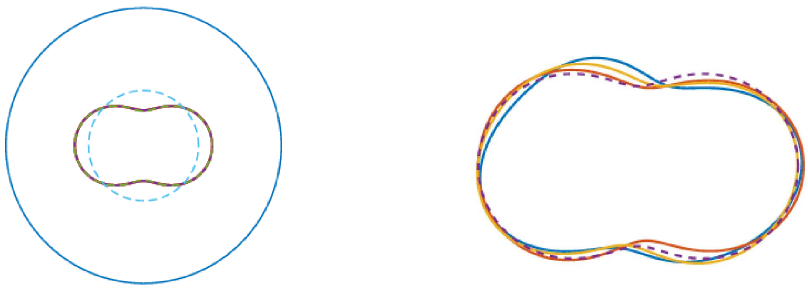
The Cauchy data on Λ were generated by solving the direct problem (1)-(3) for $f = 1$ on Λ and calculating g as the normal derivative on Λ. The noisy data were formed as

$$g^\delta = g + \delta(2\eta - 1)\|g\|_{L_2(\Lambda)}$$

with the noise level δ and the random value $\eta \in (0, 1)$. The results of the numerical experiments for exact and noisy data with $\delta = 5\%$ are reflected on Fig. 1. Here we used the following discretization parameters $n = 16$, $m = 4$ and $\epsilon = 0.0001$. The values of regularization parameters, numbers of iterations and L_2-errors are given in Tabl. 1.

	δ	It.	E	α	β
Algorithm 1	0%	7	0.00561	10^{-13}	10^{-5}
	5%	8	0.07367	10^{-10}	10^{-3}
Algorithm 2	0%	21	0.00614		10^{-2}
	5%	17	0.03843		10^{-1}
Algorithm 3	0%	21	0.00322	10^{-14}	10^{-7}
	5%	15	0.04714	10^{-5}	10^{-1}

TABL. 1. Errors and regularization parameters



a). Reconstruction for the exact data b). Reconstruction for 5% noise in the data

FIG. 1. Reconstruction of the boundary Γ

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ON THE GENERALIZED SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM WITH NEUMANN CONDITION FOR THE WAVE EQUATION BY THE USE OF THE RETARDED DOUBLE LAYER POTENTIAL AND THE LAGUERRE TRANSFORM

S. V. Litynskyy, A. O. Muzychuk

Резюме. Описано і обгрунтовано підхід до розв'язування мішаної задачі Неймана для однорідного хвильового рівняння, який базується на інтегральному перетворенні Лагера за часовою змінною і граничних інтегральних рівняннях. Для подання узагальненого розв'язку такої задачі використано запізнюючий потенціал подвійного шару, густину якого шукають у вигляді ряду Фур'є-Лагера. Для коефіцієнтів розвинення отримано аналітичні формули. В результаті вихідну нестаціонарну задачу зведено до еквівалентної послідовності граничних інтегральних рівнянь.

Abstract. Approach for solving of the initial-boundary value problem for the homogeneous wave equation with the Neumann condition is described and proved. It is based on the Laguerre transform in the time domain and the boundary integral equations. The retarded double layer potential is used for representation of generalized solution of such problem in some weighted Sobolev spaces. The density of retarded potential is expanded in Fourier-Laguerre series, coefficients of which have special convolution form. As a result, the initial-boundary value problem is reduced to an equivalent sequence of boundary integral equations.

1. Introduction

Retarded surface potentials are useful tools for the integral representation of generalized solutions of initial-boundary value problems for the wave equation with homogeneous initial conditions [1, 2, 6]. Their advantages in applications are, first of all, caused by the generality of domain form. In addition, they allow to reduce initial-boundary value problems to equivalent time-dependent boundary integral equations (TDBIEs, also known as retarded potential boundary integral equations), with unknown densities of potentials that are determined at each moment of time only on the domain's boundary [7, 12, 17]. Further, they implicitly impose radiation conditions at infinity.

However, practical usage of retarded potentials has some computational complexity, caused by the presence of dependency of potential density on the time and the spatial coordinates (so-called delay, see for example [7]). To overcome

Key words. Boundary integral equation method; wave equation; Sobolev spaces; generalized solution; retarded surface potentials; Laguerre transform; time domain boundary integral equations.

such problems, the following approaches have been used: one of traditional discretisations on spatial variables is applied to unknown values and auxiliary problems are used for calculation of the time dependency. In particular, a convolution quadrature [17] method has been utilized in many applications. It is based on the use of sustainable methods for ordinary differential equations. Using this method in the time is more stable than using Galerkin or collocation time approximations.

Another way to take account of dependence in the time domain is the Fourier-Laplace integral transform over the time variable [1, 6, 7]. This method is well suitable for theoretical investigations, however, it is complex (except for some cases) to perform corresponding inverse transform in applications. In this respect the Laguerre transform, for which the inverse transform is to find the sum of corresponding Fourier-Laguerre series, proved to be more constructive. In combination with the method of boundary integral equations (BIEs) such transform was used in [3, 8, 10, 13, 15, 18, 21] for numerical solution of various evolution problems.

In [16] we considered the generalized solution of the Dirichlet initial-boundary value problem for the wave equation with homogeneous initial conditions. Its representation was built by using the retarded single layer potential in some weighted Sobolev spaces, in which the desired solution and the potential density allow the Fourier-Laguerre expansion over the time. In this case the Fourier-Laguerre coefficients for the potential density are defined as solutions of the BIEs. This work is concerned with applying the same method to the analogical problem for the wave equation but with the Neumann boundary condition. In this case we deal with the retarded double layer potential.

We begin in Section 2 with a brief description of the proposed method. Section 3 contains the basic definitions of proper functional spaces, followed by a formulation of the main theorem about conditions under which the generalized solution of the problem belongs to the desired weighted Sobolev spaces and can be obtained by the proposed method. In Section 4 we investigate the regularity of the retarded double layer potential depending on the smoothness of its density. Definitions of the Laguerre transform and a q-convolution of sequences are introduced in Section 5, as well as the Fourier-Laguerre expansion is given for the potential's density and the representation formula for the corresponding Fourier-Laguerre coefficients are obtained. In Section 6 we explain how this approach leads to a sequence of BIE, solutions of which are Fourier-Laguerre coefficients of the unknown potential's density. At the end we prove a theorem that has been referred to above.

2. Reduction of the Neumann problem to a sequence of BIE

Let Ω be a domain in $\mathbb{R}^3$ with Lipschitz boundary Γ, $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$, $\mathbb{R}_+ := (0, \infty)$, $Q := \Omega \times \mathbb{R}_+$, $\Sigma := \Gamma \times \mathbb{R}_+$, and $\boldsymbol{\nu}(x)$ be a unit vector in the direction of the outward normal to the surface Γ at a point $x \in \Gamma$.

Let us consider the initial-boundary value problem: find a function $u(x, t)$, $(x, t) \in \overline{Q}$, that satisfies (in some sense) the homogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = 0, \quad (x,t) \in Q, \tag{1}$$

homogeneous initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in \Omega, \tag{2}$$

and the Neumann boundary condition

$$\partial_{\boldsymbol{\nu}(x)} u(x,t) = g(x,t), \quad (x,t) \in \Sigma. \tag{3}$$

Here $\Delta = \sum\limits_{i=1}^{3} \partial^2/\partial x_i^2$ is the Laplace operator and $\partial_{\boldsymbol{\nu}}$ denotes the normal derivative operator. Note that for a sufficiently smooth function u and the surface Γ operator $\partial_{\boldsymbol{\nu}}$ can be expressed as

$$\partial_{\boldsymbol{\nu}(x)} u(x,\cdot) = \boldsymbol{\nu}(x) \cdot \nabla_x u(x,\cdot),$$

where ∇_x is the gradient operator.

We use the retarded double layer potential to solve the problem (1)-(3)

$$(\mathcal{D}\lambda)(x,t) := \frac{1}{4\pi} \int\limits_{\Gamma} \boldsymbol{\nu}(y) \cdot \nabla_y \left(\frac{\lambda(z, t - |x-y|)}{|x-y|} \right) \bigg|_{z=y} d\Gamma_y, \quad (x,t) \in Q, \tag{4}$$

where $\lambda : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a density. It is known (see, e.g., [1], [21]) that if an arbitrary function $\lambda(y,\tau)$, $(y,\tau) \in \Gamma \times \mathbb{R}$, is smooth enough and $\lambda(y,\tau) = 0$ when $y \in \Gamma, \tau \leq 0$, then function

$$u(x,t) := (\mathcal{D}\lambda)(x,t), \quad (x,t) \in Q, \tag{5}$$

satisfies (in classic sense) the wave equation and initial conditions. In order for the function u to satisfy the boundary conditions (3) we will consider the following limit

$$(\mathcal{W}\lambda)(x,t) := \frac{1}{4\pi} \boldsymbol{\nu}(x) \cdot \lim_{x' \to x} \nabla_{x'} \int\limits_{\Gamma} \boldsymbol{\nu}(y) \cdot \nabla_y \left(\frac{\lambda(z, t - |x'-y|)}{|x'-y|} \right) \bigg|_{z=y} d\Gamma_y, \tag{6}$$

where $x' := x - \epsilon \boldsymbol{\nu}(x) \in \Omega$, $\epsilon > 0$ notes a point close to the points $x \in \Gamma$, understanding approach of $x' \to x$ by $\epsilon \to 0$. The function u satisfies the boundary condition (3), if the function λ is a solution of the TDBIE

$$(\mathcal{W}\lambda)(x,t) = g(x,t), \quad (x,t) \in \Sigma. \tag{7}$$

To find the solution of the equation (7) we use the Laguerre transform, namely the expansion of function in the Fourier-Laguerre series by Laguerre polynomials $\{L_j(\sigma\cdot)\}_{j\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}$ is a set of natural numbers and $\sigma > 0$ is a parameter. It is known (see, e.g., [11]) that the system of Laguerre polynomials forms an orthogonal basis in the space $L^2_\sigma(\mathbb{R}_+)$ $v : \mathbb{R}_+ \to \mathbb{R}$ of functions such that $\int\limits_{\mathbb{R}_+} |v(\tau)|^2 e^{-\sigma\tau} d\tau < \infty$, therefore, $v(\tau) = \sum\limits_{j=0}^{\infty} v_j\, L_j(\sigma\tau)$, $\tau \in \mathbb{R}_+$, where $v_j := \sigma \int\limits_{\mathbb{R}_+} v(\tau)\, L_j(\sigma\tau)\, e^{-\sigma\tau} d\tau$ $(j \in \mathbb{N}_0)$ are the Laguerre-Fourier coefficients of function v.

Therefore, the solution of the TDBIE (7) can be expressed as:

$$\lambda(y,\tau) = \begin{cases} \sum\limits_{j=0}^{\infty} \lambda_j(y)\, L_j(\sigma\tau), & y \in \Gamma,\ \tau \in \mathbb{R}_+, \\ 0, & y \in \Gamma,\ \tau \in \mathbb{R} \setminus \mathbb{R}_+, \end{cases} \tag{8}$$

where λ_j ($j \in \mathbb{N}_0$) are the corresponding Laguerre-Fourier coefficients of the unknown function λ. In the case of the retarded argument with arbitrary value $a > 0$ we have an expansion

$$\lambda(y, t-a) = \sum_{j=0}^{\infty} \tilde{\lambda}_j(y,a)\, L_j(\sigma t), \tag{9}$$

where coefficients $\tilde{\lambda}_j(y,a)$ have the representation formula, that was obtained in [16]

$$\tilde{\lambda}_j(y,a) = e^{-\sigma a} \sum_{i=0}^{j} \zeta_{j-i}(\sigma a)\, \lambda_i(y), \quad j \in \mathbb{N}_0, \tag{10}$$

and where

$$\zeta_0(s) := 1, \ \ \zeta_k(s) := L_k(s) - L_{k-1}(s), \ \ s \in \overline{\mathbb{R}_+} = [0,\infty), \quad k \in \mathbb{N}. \tag{11}$$

Then, taking into account (9) and (10), we will have

$$\begin{aligned} \lambda(y, t-|x-y|) = & e^{-\sigma|x-y|} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \zeta_{j-i}(\sigma|x-y|)\, \lambda_i(y) \right) L_j(\sigma t), \\ & x, y \in \Gamma,\ t \in \mathbb{R}_+, \end{aligned} \tag{12}$$

and then introducing notation similar to (6)

$$(W_k\xi)(x) := \frac{1}{4\pi} \boldsymbol{\nu}(x) \cdot \lim_{x' \to x} \nabla_{x'} \int\limits_{\Gamma} \xi(y) \boldsymbol{\nu}(y) \cdot \nabla_y e_k(x'-y) d\Gamma_y, \tag{13}$$

where

$$e_k(z) := (4\pi|z|)^{-1} \zeta_k(\sigma|z|) e^{-\sigma|z|} \ \text{ at } \ z \in \mathbb{R}^3 \setminus \{0\}, \ \ k \in \mathbb{N}_0, \tag{14}$$

for the normal derivative of the retarded double layer potential (6) we obtain an expansion

$$(\mathcal{W}\lambda)(x,t) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} (W_{j-i}\lambda_i)(x) \right) L_j(\sigma t), \quad x, y \in \Gamma,\ t \in \mathbb{R}_+. \tag{15}$$

Now lets write the Fourier-Laguerre expansion of the function g

$$g(x,t) = \sum_{j=0}^{\infty} g_j(x)\, L_j(\sigma t), \quad (x,t) \in \Sigma, \tag{16}$$

where $g_j(x) = \sigma \int\limits_{\mathbb{R}_+} g(x,\tau)\, L_j(\sigma\tau)\; e^{-\sigma\tau} d\tau,\ x \in \Gamma,\ j \in \mathbb{N}_0$. Taking into account (15) and (16) along with (7) and equating expressions near the Laguerre polynomials with the same indexes, we get an infinite triangular system of BIE for finding the Laguerre-Fourier coefficients $\lambda_0,\ \lambda_1, ..., \lambda_j, ...$ of the density λ

$$\sum_{i=0}^{j}(W_{j-i}\lambda_i)(x) = g_j(x), \quad x \in \Gamma,\ j \in \mathbb{N}_0. \tag{17}$$

It is easy to see that system (17) can be rewritten as a recursive sequence of equations

$$\begin{cases} (W_0\lambda_0)(x) = g_0(x), \\ (W_0\lambda_1)(x) = \tilde{g}_1(x), \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ (W_0\lambda_j)(x) = \tilde{g}_j(x), \quad j \in \mathbb{N}, \quad x \in \Gamma, \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{cases} \tag{18}$$

where

$$\tilde{g}_j(x) := g_j(x) - \sum_{i=0}^{j-1}(W_{j-i}\lambda_i)(x), \quad j \in \mathbb{N}. \tag{19}$$

For every $j \in \mathbb{N}_0$ the corresponding j-th equation (18) is hypersingular equation that has the form

$$(W_0\xi)(x) = h(x), \quad x \in \Gamma. \tag{20}$$

It is known [4,9] that the equation (20) has a unique solution ξ for an arbitrary function h within a fairly broad class. To find the solution of this equation one can use numerical methods (see for example [24] and references there).

After finding the solution $\lambda_0,\ \lambda_1, ...$ of the BIE system (17) (same as a solution of the sequence (18)), the generalized solution of the problem (1)-(3) can be presented using (4), (5) and (12) as a sum of the series

$$u(x,t) = \frac{1}{4\pi}\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\int_{\Gamma}\lambda_i(y)\,\boldsymbol{\nu}(y)\cdot\nabla_y e_{j-i}(x-y)d\Gamma_y\right)L_j(t), \ \ (x,t) \in Q. \tag{21}$$

If we introduce a notation

$$(D_k\xi)(x) := \frac{1}{4\pi}\int_{\Gamma}\xi(y)\boldsymbol{\nu}(y)\cdot\nabla_y e_k(x'-y)d\Gamma_y, \tag{22}$$

the formula (21) can be rewritten as:

$$u(x,t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}D_{j-i}\lambda_i(x)\right)L_j(\sigma t), \ \ (x,t) \in Q. \tag{23}$$

If there exists a sum of the series (23) we can consider its partial sum as an approximate solution for the problem (1)-(3). In this case one can choose (by some criteria) value N and find from the system (18) the first components $\lambda_0,\ \lambda_1, ..., \lambda_N$ of its solution. Then the approximate solution of the problem (1)-(3) is the partial sum

$$\widetilde{u}_N(x,t) = \sum_{j=0}^{N}\left(\sum_{i=0}^{j}D_{j-i}\lambda_i(x)\right)L_j(\sigma t), \ \ (x,t) \in Q. \tag{24}$$

We can use the representation (24) for the numerical solution of the problem (1)-(3).

3. Variational formulation of the problem (1)-(3)

First, we need to introduce some additional notations. Let $L^2(\Omega)$ be the Lebesgue space of square integrable functions $v : \Omega \to \mathbb{R}$ with inner product

$$(v, w)_{L^2(\Omega)} := \int\limits_{\Omega} vw dx, \;\; v, w \in L^2(\Omega),$$

and norm $\|v\|_{L^2(\Omega)} := \sqrt{(v, v)_{L^2(\Omega)}}$, and $H^1(\Omega)$ be the Sobolev space of functions $v \in L^2(\Omega)$, having generalized derivatives of $v_{x_1}, v_{x_2}, v_{x_3}$ in $L^2(\Omega)$, with inner product

$$(v, w)_{H^1(\Omega)} := \int\limits_{\Omega} (\nabla v \nabla w + vw)\, dx, \;\; v, w \in H^1(\Omega),$$

and norm $\|v\|_{H^1(\Omega)} := \sqrt{(v, v)_{H^1(\Omega)}}$, $v \in H^1(\Omega)$. Let us denote $H^{1/2}(\Gamma)$ a space of traces of elements of $H^1(\Omega)$ on the surface Γ, $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ a trace operator, $H^{-1/2}(\Gamma) := \big(H^{1/2}(\Gamma)\big)'$ a conjugate to $H^{1/2}(\Gamma)$ space, and $< \cdot, \cdot >_\Gamma$ a duality relation for $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Also let $H^1_0(\Omega)$ be a closure of the space $C^\infty_0(\Omega)$ with norm $\|\cdot\|_{H^1(\Omega)}$ and $H^{-1}(\Omega) := (H^1_0(\Omega))'$ be the conjugate to $H^1_0(\Omega)$ space. In the space $H^1(\Omega)$ we also consider a subspace $H^1(\Omega, \Delta) := \big\{\; v \in H^1(\Omega) \,|\, \Delta v \in L^2(\Omega)\big\}$ with the norm

$$\|v\|_{H^1(\Omega,\Delta)} := \Big(\|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

Let X be a Hilbert space with inner product $(\cdot, \cdot)_X$ and inducted norm $||\cdot||_X$. For some parameter $\sigma > 0$ we consider a weighted Lebesgue space $L^2_\sigma(\mathbb{R}_+; X)$ [5] with weight $\rho_\sigma(t) = e^{-\sigma t}$, $t \in \mathbb{R}_+$, elements of which are measurable functions $v : \mathbb{R}_+ \to X$ such that $\int\limits_{\mathbb{R}_+} ||v(t)||^2_X\, e^{-\sigma t} dt < \infty$. This space is equipped with inner product

$$(v, w)_{L^2_\sigma(\mathbb{R}_+;X)} := \int\limits_{\mathbb{R}_+} \big(v(t), w(t)\big)_X\, e^{-\sigma t} dt, \;\; v, w \in L^2_\sigma(\mathbb{R}_+; X), \tag{25}$$

and the norm

$$\|v\|_{L^2_\sigma(\mathbb{R}_+;X)} := \sqrt{(v, v)_{L^2_\sigma(\mathbb{R}_+;X)}}, \;\; v \in L^2_\sigma(\mathbb{R}_+; X). \tag{26}$$

Note that the space $L^2_\sigma(\mathbb{R}_+; X)$ is complete [22, section II.1]. We will assume that the elements of space $L^2_\sigma(\mathbb{R}_+; X)$ are extended with zero for non-positive arguments.

For any $m \in \mathbb{N}$ let us denote the weighted Sobolev space as

$$H^m_\sigma(\mathbb{R}_+; X) := \{\;\; v \in L^2_\sigma(\mathbb{R}_+; X) \,|\, v^{(k)} \in L^2_\sigma(\mathbb{R}_+; X), \; k = \overline{1, m}\} \tag{27}$$

with norm

$$\|v\|_{H^m_\sigma(\mathbb{R}_+;X)} := \left(\sum_{k=0}^{m} \left\|v^{(k)}\right\|^2_{L^2_\sigma(\mathbb{R}_+;X)}\right)^{1/2}. \tag{28}$$

Here derivatives $v^k (k \in \mathbb{N})$ are understood in terms of the space $\mathcal{D}'(\mathbb{R}_+;X)$, elements of which are distributions with values in the space X. Note that $H^1_\sigma(\mathbb{R}_+;X) \subset C(\overline{\mathbb{R}_+};X)$ [5, theorem 7, section XVIII].

Let us also denote following spaces:

$$L^2_{\text{loc}}(\overline{\mathbb{R}_+};X) := \{v : \mathbb{R}_+ \to X - \text{ measurable} \mid \ ||v(\cdot)||_X \in L^2(0,\tau) \ \ \forall \tau > 0\},$$

$$H^1_{\text{loc}}(\overline{\mathbb{R}_+};X) := \{v \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};X) | \ \ v' \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};X)\}.$$

Definition 1. Let $g \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};H^{-1/2}(\Gamma))$. A generalized solution of the problem (1)-(3) is a function $u \in H^1_{\text{loc}}(\overline{\mathbb{R}_+};L^2(\Omega)) \cap L^2_{\text{loc}}(\overline{\mathbb{R}_+};H^1(\Omega))$, which satisfies the first of the initial conditions (2) and the integral identity

$$\iint\limits_{Q} (\nabla u \nabla v - u'v')dxdt = \iint\limits_{\Sigma} g\gamma_0 v d\Gamma dt \tag{29}$$

for any $v \in H^1(\mathbb{R}_+;L^2(\Omega)) \cap L^2(\mathbb{R}_+;H^1(\Omega))$ such that $\operatorname{supp} v$ is a bounded set.

Note that there exists at most one generalized solution of the problem (1)-(3) [19, Theorem 1, Ch. V, §2].

We introduce a couple more notations. As the sequence of elements of set X we understand mapping $\mathbf{V} : \mathbb{N}_0 \to X$ (denoted by **bold** letter) and write it as a vector-column $\mathbf{v} := (v_0, v_1, ...)^\top$. All possible sequences of elements of the set X are denoted by X^∞. It is clear that when X is a linear space, then X^∞ is also a linear space. Recall that

$$l^2 := \left\{\mathbf{v} \in \mathbb{R}^\infty \mid \quad \sum_{j=0}^{\infty} |v_j|^2 < +\infty\right\}$$

with the inner product $(\mathbf{v},\mathbf{w}) = \sum\limits_{j=0}^{\infty} v_j w_j, \ \ \mathbf{v},\mathbf{w} \in l^2$ and the norm $\|\mathbf{v}\|_{l^2} := \left(\sum\limits_{j=0}^{\infty} |v_j|^2\right)^{1/2}, \ \ \mathbf{v} \in l^2$.

Let X be a Hilbert space with inner product $(\cdot,\cdot)_X$ and inducted norm $||\cdot||_X$. We consider the Hilbert space

$$l^2(X) := \left\{\mathbf{v} \in X^\infty \mid \quad \sum_{j=0}^{\infty} \|v_j\|^2_X < +\infty\right\}$$

with the inner product $(\mathbf{v},\mathbf{w}) = \sum\limits_{j=0}^{\infty} (v_j,w_j)_X, \ \mathbf{v},\mathbf{w} \in l^2(X)$ and the norm $||\mathbf{v}||_{l^2(X)} := \left(\sum\limits_{j=0}^{\infty} \|v_j\|^2_X\right)^{1/2}, \ \mathbf{v} \in l^2(X)$. It is obvious that $l^2 = l^2(\mathbb{R})$.

Definition 2 ([14])**.** Let X, Y, Z be arbitrary sets and $q : X \times Y \to Z$ be some mapping. By a *q-convolution* of sequences $\mathbf{u} \in X^\infty$ and $\mathbf{v} \in Y^\infty$ we understand the sequence $\mathbf{w} := (w_0, w_1, ..., w_j, ...)^\top \in Z^\infty$, whose elements are obtained by the rule

$$w_j := \sum_{i=0}^{j} q\left(u_{j-i}, v_i\right) \equiv \sum_{i=0}^{j} q\left(u_i, v_{j-i}\right), \quad j \in \mathbb{N}_0; \tag{30}$$

the q-convolution of $\mathbf{u}$ and $\mathbf{v}$ is shortly written in the form $\mathbf{w} = \mathbf{u} \underset{q}{\circ} \mathbf{v}$.

Let $X = \mathbb{R}$ and $Y = Z$ be linear spaces and $q(u, v) := uv,\ u \in \mathbb{R},\ v \in Y$. Then the components of q-convolution of arbitrary $\mathbf{u} \in \mathbb{R}^\infty$ and $\mathbf{v} \in Y^\infty$ will be denoted as

$$w_j = \sum_{i=0}^{j} u_{j-i} v_i, \quad j \in \mathbb{N}_0, \tag{31}$$

and the q-convolution would be denoted as $\mathbf{w} := \mathbf{u} \underset{\mathbb{R}\times Y}{\circ} \mathbf{v}$.

If $X = H^{-1/2}(\Gamma),\ Y = H^{1/2}(\Gamma),\ Z = \mathbb{R}$ and $q(u, v) :=< u, v >_\Gamma,\ u \in H^{-1/2}(\Gamma),\ v \in H^{1/2}(\Gamma)$, for components of the q-convolution of arbitrary sequences $\mathbf{u} \in \left(H^{-1/2}(\Gamma)\right)^\infty$ and $\mathbf{v} \in \left(H^{1/2}(\Gamma)\right)^\infty$ we will have

$$w_j = \sum_{i=0}^{j} < u_{j-i}, v_i >_\Gamma, \quad j \in \mathbb{N}_0, \tag{32}$$

and will write $\mathbf{w} := \mathbf{u} \underset{\Gamma}{\circ} \mathbf{v}$.

Another example concerns the q-convolutions of linear operators when $X = \mathcal{L}(Y, Z)$ is the space of linear operators acting from the space Y into the space Z and $q(A, v) := Av$, $A \in \mathcal{L}(Y, Z),\ v \in Y$, for components of the q-convolution of arbitrary sequences $\mathbf{A} \in \left(\mathcal{L}(Y, Z)\right)^\infty$ and $\mathbf{v} \in Y^\infty$ we will have the following formula

$$w_j = \sum_{i=0}^{j} A_{j-i} v_i, \quad j \in \mathbb{N}_0, \tag{33}$$

and will write $\mathbf{w} := \mathbf{A} \underset{Z}{\circ} \mathbf{v}$.

Based on the above, we define the sequence

$$\mathbf{u}(x) = \big(\mathbf{D} \underset{H^1(\Omega)}{\circ} \boldsymbol{\lambda}\big)(x), \quad x \in \Omega, \tag{34}$$

which is the q-convolution of the sequence $\mathbf{D}$ composed of operators $D_k : H^{1/2}(\Gamma) \to H^1(\Omega))$, $k \in \mathbb{N}_0$, given by the formula (22), and the sequence $\boldsymbol{\lambda}$ of Fourier-Laguerre coefficients of the function λ. Similarly, BIE system (17) can be rewritten as

$$\mathbf{W} \underset{H^{-1/2}(\Gamma)}{\circ} \boldsymbol{\lambda} = \mathbf{g} \quad \text{in} \quad l^2(H^{-1/2}(\Gamma)), \tag{35}$$

where $\mathbf{W} : l^2(H^{1/2}(\Gamma)) \to l^2(H^{-1/2}(\Gamma))$ is a boundary operator whose components act in accordance with (13), and $\mathbf{g}$ is the sequence of Fourier-Laguerre coefficients of the function g.

Now we can formulate the main result of this paper as the following statement.

Theorem 1. *Let $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then there exists a unique generalized solution of the problem (1)-(3), it belongs to the space $H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$ and for any $\sigma \geq \sigma_0$ such an inequality holds*

$$||u||_{H^{m+1}_{\sigma}(\mathbb{R}_+;H^1(\Omega))} \leq C||g||_{H^{m+4}_{\sigma}(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{36}$$

where $C > 0$ is a constant that is not dependent on g.

In addition, the generalized solution of the problem (1)-(3) can be represented as the sum of a serie (23), that is convergent in the space $L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega, \Delta))$, where $u_j \in H^1(\Omega, \Delta)$ ($j \in \mathbb{N}_0$) are the corresponding components of the q-convolution (34), and elements of the sequence $\boldsymbol{\lambda} \in l^2(H^{1/2}(\Gamma))$ are solutions of BIE system (35), in which $\mathbf{g} \in l^2(H^{-1/2}(\Gamma))$ is the sequence of Laguerre-Fourier coefficients for the function g.

Proof of Theorem 1 will be presented further on.

4. Some properties of the retarded double layer potential

For examination of the generalized solution of the problem (1)-(3) we need some results of the work [1].

Proposition 1 ([1], Theorem 1)**.** *Let $g \in H^1_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$. Then unique generalized solution of the problem space (1)-(3) exists, it belongs to space*

$$H^1_{\sigma_0}(\mathbb{R}_+; L^2(\Omega)) \cap L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$$

and the following inequality holds:

$$||u||_{L^2_{\sigma}(\mathbb{R}_+;H^1(\Omega))} + ||u'||_{L^2_{\sigma}(\mathbb{R}_+;L^2(\Omega))} \leq C_1||g||_{H^1_{\sigma}(\mathbb{R}_+;H^{-1/2}(\Gamma))} \ \ \forall \sigma \geq \sigma_0, \tag{37}$$

where $C_1 > 0$ is a constant.

In addition, the generalized solution of the problem (1)-(3) can be represented as a retarded double layer potential $\mathcal{D}\lambda$ with density $\lambda \in L^2_{\sigma}(\mathbb{R}_+; H^{1/2}(\Gamma))$,

$$||\lambda||_{L^2_{\sigma}(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq C_2||g||_{H^1_{\sigma}(\mathbb{R}_+;H^{-1/2}(\Gamma))} \ \ \forall \sigma \geq \sigma_0, \tag{38}$$

where $C_2 > 0$ is a constant.

Let us outline the proof of the statement 1, received results will be exploited further for the proof of 1.

First, consider some auxiliary spaces. Let X be arbitrary Banach space with a norm $|| \cdot ||_X$. By $\mathcal{D}'(\mathbb{R}; X)$ we denote the space of distributions with values in the space X and by $\mathcal{D}'_+(\mathbb{R}; X)$ we denote the space of so-called causal distributions, consisting of distributions $v \in \mathcal{D}'(\mathbb{R}; X)$, for which the condition $\langle v, \phi\rangle = 0$ holds for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\phi \subset (-\infty, 0)$. For any $\sigma_0 > 0$ let us define a space

$$\mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X) := \{ \ f \in \mathcal{D}'_+(\mathbb{R}; X) \mid e^{-\sigma_0 \cdot} f(\cdot) \in \mathcal{S}'_+(\mathbb{R}; X) \ \},$$

where $\mathcal{S}'_+(\mathbb{R}; X)$ denotes the space of slow casual distributions.

Note that for slow casual distributions one can define the Fourier transform over the time variable (See, e.g., [5, section XVI, §2, definition 7])

$$\mathcal{F} : \mathcal{S}'_+(\mathbb{R}; X) \to \mathcal{S}'_+(\mathbb{R}; X). \tag{39}$$

It is an isomorphic mapping from $\mathcal{S}'_+(\mathbb{R}; X)$ onto $\mathcal{S}'_+(\mathbb{R}; X)$ and enables us to define the Fourier-Laplace transform for any element $f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X)$ [5, section XVI, §2, definition 8]:

$$\widehat{F}(\omega) := \mathcal{F}(e^{-\sigma\cdot} f(\cdot))(\eta), \;\; \omega = \eta + i\sigma \in \mathbb{R} \times (\sigma_0, +\infty). \tag{40}$$

In case of $f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X) \cap L^1_{\text{loc}}(\mathbb{R}_+; X)$ this transform has an integral representation

$$\widehat{f}(\omega) := \int_{\mathbb{R}} e^{i\eta t} e^{-\sigma t} f(t) dt = \int_{\mathbb{R}} e^{i\omega t} f(t) dt, \;\; \omega = \eta + i\sigma \in \mathbb{R} \times (\sigma_0, +\infty). \tag{41}$$

As we can see the Fourier-Laplace transform is applicable to the elements of functional spaces that appear in the definition of the generalized solution u of the problem (1)-(3). So with its help the initial-boundary value problem (1)-(3) can be reduced to following boundary value problem regarding a function $\widehat{u}(\cdot, \omega) \in H^1(\Omega, \Delta)$:

$$\Delta\widehat{u} + \omega^2\widehat{u} = 0 \;\text{ in } \Omega, \tag{42}$$

$$\gamma_1\widehat{u} = \widehat{g} \text{ on } \Gamma, \tag{43}$$

where $\widehat{g}(\cdot, \omega) \in H^{-1/2}(\Gamma)$ is a known function and $\omega \in \mathbb{R} \times (\sigma_0, +\infty)$ is a parameter.

Solution of the problem (42), (43) can be represented as a double layer potential

$$\widehat{u}(x, \omega) = \big(\widehat{D}_\omega\widehat{\lambda}\big)(x) := \frac{1}{4\pi} \int_\Gamma \widehat{\lambda}(y, \omega)\, \boldsymbol{\nu}(y) \cdot \nabla_y \frac{e^{i\omega|x-y|}}{|x-y|}\, d\Gamma_y, \;\; x \in \Omega, \tag{44}$$

whose density $\widehat{\lambda}(\cdot, \omega) \in H^{1/2}(\Gamma)$ is a solution of BIE

$$\widehat{W}_\omega\widehat{\lambda} = \widehat{g} \text{ in } H^{-1/2}(\Gamma), \tag{45}$$

where $\widehat{W}_\omega := \gamma_1 \circ \widehat{D}_\omega$. A boundary operator $\widehat{W}_\omega$ is $H^{1/2}$-elliptical on Γ, that implies the existence and uniqueness of the solution for BIE(45).

The integral (44) exists because of $\widehat{\lambda}(\cdot, \omega) \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and $\frac{e^{i\omega|x-y|}}{|x-y|}$ is an infinitely differentiable function for an arbitrary fixed point $x \in \Omega$. In addition, according to the [4, Theorem 1], the double layer potential and its normal derivative are bounded operators, respectively, $\widehat{D}_\omega : H^{1/2}(\Gamma) \to H^1(\Omega, \Delta)$ and $\widehat{W}_\omega : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$.

As we see, the boundary value problem (42), (43) and BIE (45) depend on parameter ω, consequently, their solutions, accordingly, $\widehat{u}(\cdot, \omega)$ and $\widehat{\lambda}(\cdot, \omega)$, and the double layer potential $\widehat{D}_\omega$ and the boundary operator $\widehat{W}_\omega$ can be considered as functions of parameter ω. They are proved to be holomorphic in half-space $\mathbb{R} \times (\sigma_0, +\infty)$ and satisfy following estimates [1, inequality (2.6),(2.7) and (2.11)], [23, inequality (3.17) and (3.18)]:

$$||\widehat{u}(\cdot, \omega)||_{H^1(\Omega)} \le \tilde{C}_1|\omega|||\widehat{g}(\cdot, \omega)||_{H^{-1/2}(\Gamma)}, \tag{46}$$

$$||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)} \leq \tilde{C}_2|\omega|||\widehat{g}(\cdot,\omega)||_{H^{-1/2}(\Gamma)}, \tag{47}$$

$$||\widehat{W}_\omega\widehat{\lambda}||_{H^{-1/2}(\Gamma)} \leq \tilde{C}_3|\omega|^2\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{48}$$

$$||\widehat{D}_\omega\widehat{\lambda}||_{H^1(\Omega)} \leq \tilde{C}_4|\omega|^{3/2}\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{49}$$

$$||\widehat{D}_\omega\widehat{\lambda}||_{H^1(\Omega,\Delta)} \leq \tilde{C}_5|\omega|^{5/2}\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{50}$$

where $\tilde{C}_i > 0$ are some constants.

Proposition 2 ([5], section XVI, §2, Theorem 1)**.** *Let X be a Banach space over the field $\mathbb{C}$ of complex numbers with norm $||\cdot||_X$, and $\omega \mapsto \widehat{f}(\omega)$ be a function defined in $\mathbb{C}$ with values in the space X. For the function $\widehat{f}(\omega)$ to be the Fourier-Laplace transform of the distribution $f \in \mathcal{D}'(\mathbb{R};X)$ with support* $\operatorname{supp} f \subset [\alpha,+\infty)$ *it is necessary and sufficient that $\widehat{f}(\omega)$ is holomorphic in the half-space $\mathbb{R}\times(\sigma_0,+\infty)$ with values in X and satisfies inequality*

$$||\widehat{f}(\omega)||_X \leq e^{-\sigma\alpha}\mathrm{Pol}(|\omega|),\ \ \omega = \eta + i\sigma \in \mathbb{R}\times(\sigma_0,+\infty), \tag{51}$$

where $\mathrm{Pol}(|\omega|)$ *is a polynom of the variable $|\omega|$.*

By the statement 2 one can prove from inequalities (46)-(50) the existence of distributions that match the generalized solution of the problem (1)-(3), retarded double layer potential and its density. They are elements of spaces $\mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)$ with values in the appropriate space X (see e.g. [1, Theorem 1], and [6, section 2]) such that

$$\widehat{\mathcal{D}\lambda} = \widehat{D}_\omega\widehat{\lambda} \ \text{ and } \ \widehat{\mathcal{W}\lambda} = \widehat{W}_\omega\widehat{\lambda}.$$

Using inequalities (46)-(50) we can easily get estimates of the generalized solution of the problem (1)-(3), and the retarded double layer potential. To do this, let us consider in the set $\mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)$ for arbitrary values $\sigma \geq \sigma_0$ and $p \in \mathbb{R}$ a space

$$\mathcal{H}^p_\sigma(\mathbb{R}_+;X) := \{\ f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)\,|\int\limits_{\mathbb{R}+i\sigma}|\omega|^{2p}\,||\widehat{f}(\omega)||^2_X d\omega < +\infty\} \tag{52}$$

with the norm

$$||f||_{\mathcal{H}^p_\sigma(\mathbb{R}_+;X)} := \left(\frac{1}{2\pi}\int\limits_{\mathbb{R}+i\sigma}|\omega|^{2p}\,||\widehat{f}(\omega)||^2_X d\omega\right)^{1/2}. \tag{53}$$

Proposition 3 ([2], section 3.1)**.** *Let $\sigma > 0$, $m \in \mathbb{N}_0$. A function v belongs to the space $H^m_\sigma(\mathbb{R}_+;X)$ if and only if it belongs to the space $\mathcal{H}^m_{\sigma/2}(\mathbb{R}_+;X)$.*

Note that statement 3 is the consequence of Parseval-Plancherel identity:

$$\int\limits_{\mathbb{R}} e^{-2\sigma t}\left(f(t),g(t)\right)_X dt = \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\sigma}\left(\hat{f}(\omega),\hat{g}(\omega)\right)_X d\omega. \tag{54}$$

Lemma 1. *Let $\sigma > 0$, $m \in \mathbb{N}_0$. If an arbitrary function λ is an element of the space $H^{m+2}_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+;H^1(\Omega))$. If $\lambda \in H^{m+3}_\sigma(\mathbb{R}_+;$ $H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ and $\mathcal{W}\lambda \in H^m_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$.*

Proof. Let us show that for any fixed values of $p \in \mathbb{R}$ and $\alpha > 0$ the operator

$$\mathcal{D} : \mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^{p}(\mathbb{R}_+; H^{1}(\Omega)) \tag{55}$$

is bounded. To achieve this, for an arbitrary function $\lambda \in \mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+; H^{1/2}(\Gamma))$, $\alpha \geq \alpha_0$, taking into account norm definition (53) and inequality (49), following estimate can be performed:

$$\begin{aligned}
||\mathcal{D}\lambda||^2_{\mathcal{H}_\alpha^{p}(\mathbb{R}_+;H^1(\Omega))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{\mathcal{D}\lambda}||^2_{H^1(\Omega)}d\omega = \\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{D}(\cdot,\omega)\widehat{\lambda}(\cdot,\omega)||^2_{H^1(\Omega)}d\omega \leq \\
&\leq \frac{\tilde{C}_4^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p+3}||\widehat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)}d\omega = \\
&= \tilde{C}_4^2||\lambda||^2_{\mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq \tilde{C}_4^2||\lambda||^2_{\mathcal{H}_\alpha^{p+2}(\mathbb{R}_+;H^{1/2}(\Gamma))}.
\end{aligned} \tag{56}$$

Hence, the operator (55) is bounded, and, in particular, for the values $p = m$ and $\alpha = \sigma/2$ the following operator is also bounded

$$\mathcal{D} : H_\sigma^{m+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^{m}(\mathbb{R}_+; H^{1}(\Omega)). \tag{57}$$

Similarly to the previous case, but using inequality (50), for arbitrary $p \in \mathbb{R}$ and $\alpha > 0$ it can be shown that the operator

$$\mathcal{D} : \mathcal{H}_\alpha^{p+5/2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^{p}(\mathbb{R}_+; H^{1}(\Omega,\Delta)) \tag{58}$$

is also bounded, and when $p = m$ and $\alpha = \sigma/2$ the same will apply to the operator

$$\mathcal{D} : H_\sigma^{m+3}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^{m}(\mathbb{R}_+; H^{1}(\Omega,\Delta)), \quad m \in \mathbb{N}_0, \tag{59}$$

which means $\mathcal{D}\lambda \in H_\sigma^{m}(\mathbb{R}_+; H^1(\Omega,\Delta))$. It is known [4, theorem Lemma 3.2, 1] that for elements of space $H^1(\Omega,\Delta)$ we can define linear continuous operator of normal derivative $\gamma_1 : H^1(\Omega,\Delta) \to H^{-1/2}(\Gamma)$. Therefore, in this case it is legitimate to define the composition of operators $\gamma_1 \circ \mathcal{D} =: \mathcal{W}$, for which for any $p \in \mathbb{R}$ and $\alpha > 0$ using inequality (48) following estimate can be applied:

$$\begin{aligned}
||\mathcal{W}\lambda||^2_{\mathcal{H}_\alpha^{p}(\mathbb{R}_+;H^{-1/2}(\Gamma))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{W}(\cdot,\omega)\widehat{\lambda}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)}d\omega \leq \\
\leq \frac{\tilde{C}_3^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}|\omega|^4||\widehat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)}d\omega &= \tilde{C}_3^2||\lambda||^2_{\mathcal{H}_\alpha^{p+2}(\mathbb{R}_+;H^{1/2}(\Gamma))}.
\end{aligned} \tag{60}$$

This means that the operator

$$\mathcal{W} : \mathcal{H}_\alpha^{p+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^{p}(\mathbb{R}_+; H^{-1/2}(\Gamma)) \tag{61}$$

is bounded, and when $p = m$ and $\alpha = \sigma/2$ following operator is also bounded:

$$\mathcal{W} : H_\sigma^{m+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^{m}(\mathbb{R}_+; H^{-1/2}(\Gamma)). \tag{62}$$

□

5. Application of the Laguerre transform to retarded potentials

Now let us give the definition of the Laguerre transform and outline some of its properties which we have obtained in [16]. Consider a mapping $\mathcal{L}$: $L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$, where X is Hilbert space with inner product $(\cdot,\cdot)_X$ and inducted norm $||\cdot||_X$, which operates according to the rule

$$f_k := \sigma \int\limits_{\mathbb{R}_+} f(t)\, L_k(\sigma t)\, e^{-\sigma t} dt, \quad k \in \mathbb{N}_0, \tag{63}$$

where $\{L_k(\sigma\cdot)\}_{k\in\mathbb{N}_0}$ are Laguerre polynomials, which form orthogonal basis in the space $L^2_\sigma(\mathbb{R}_+)$. We will also use the notation

$$\mathcal{L}_k f \equiv (\mathcal{L}f)(k) := f_k \;\; \forall k \in \mathbb{N}_0.$$

Note that since the function $t \mapsto ||f(t)||_X |L_k(\sigma t)| e^{-\sigma t} \in L^1(\mathbb{R}_+)$, the Bochner integral in formula (63) is convergent and its value is an element of space X.

Also consider the mapping $\mathcal{L}^{-1} : l^2(X) \to L^2_\sigma(\mathbb{R}_+; X)$, which maps an arbitrary sequence $\mathbf{h} = (h_0,\, h_1, ...,\, h_k, ...)^\top$ to a function

$$h(t) := (\mathcal{L}^{-1}\mathbf{h})(t) = \sum_{k=0}^{\infty} h_k\, L_k(\sigma t),\; t \in \mathbb{R}_+. \tag{64}$$

Proposition 4 ([16], Theorem 2)**.** *The mapping $\mathcal{L} : L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$ that maps the arbitrary function f to the sequence $\mathbf{f} = (f_0,\, f_1, ...,\, f_k, ...)^\top$ according to the formula (63), is injective and its image is the space $l^2(X)$, and*

$$\|f\|^2_{L^2_\sigma(\mathbb{R}_+;X)} = \frac{1}{\sigma}\sum_{k=0}^{\infty} ||f_k||^2_X. \tag{65}$$

In addition, for the arbitrary function $f \in L^2_\sigma(\mathbb{R}_+; X)$ we have an equality

$$\mathcal{L}^{-1}\mathcal{L}f = f, \tag{66}$$

where the mapping $\mathcal{L}^{-1} : l^2(X) \to L^2_\sigma(\mathbb{R}_+; X)$ is the inverse to $\mathcal{L}$ and maps the arbitrary sequence $\mathbf{h} = (h_0,\, h_1, ...,\, h_k, ...)^\top$ to the function h according to the formula (64).

Definition 3. Let $\sigma > 0$ and X be a Hilbert space. Mappings

$$\mathcal{L} : L^2_\sigma(\mathbb{R}_+; X) \to l^2(X) \quad \text{and} \quad \mathcal{L}^{-1} : l^2(X) \to L^2_\sigma(\mathbb{R}_+; X),$$

mentioned in theorem 4, are called, respectively, direct and inverse Laguerre transforms, and the formula (65) is an analog of the Parseval equality.

Proposition 5 ([16], Lemma 1)**.** *Let $\sigma > 0$, $a > 0$ and X be a Hilbert space with inner product $(\cdot,\cdot)_X$ and the norm $||\cdot||_X$. Then for an arbitrary function $f \in L^2_\sigma(\mathbb{R}_+; X)$ function $f(\cdot - a)$ belongs to space $L^2_\sigma(\mathbb{R}_+; X)$ too and the following equalities hold:*

$$||f(\cdot - a)||_{L^2_\sigma(\mathbb{R}_+;X)} = e^{-\frac{\sigma a}{2}} ||f(\cdot)||_{L^2_\sigma(\mathbb{R}_+;X)}, \tag{67}$$

$$\tilde{\mathbf{f}}_a = e^{-\sigma a} \boldsymbol{\zeta}(\sigma a) \underset{\mathbb{R}\times X}{\circ} \mathbf{f}, \tag{68}$$

$$f(\,\cdot - a) = e^{-\sigma a}\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\zeta_{j-i}(\sigma a)f_i\right)L_j(\sigma\cdot) \;\; in \;\; L^2_\sigma(\mathbb{R}_+;X), \tag{69}$$

where $\boldsymbol{f} = \mathcal{L}f(\cdot)$ *and* $\tilde{\mathbf{f}}_a := \mathcal{L}f(\cdot - a)$.

Using statements 4 and 5 we can outline conditions for the density λ of the retarded double layer potential $\mathcal{D}\lambda$, which guarantees that the Fourier-Laguerre expansions for this potential

$$(\mathcal{D}\lambda)(t) = \sum_{j=0}^{\infty} u_j\, L_j(\sigma t), \;\; x \in \Omega,\, t \in \mathbb{R}_+, \tag{70}$$

and its normal derivative

$$(\mathcal{W}\lambda)(x,t) = \sum_{j=0}^{\infty} \tilde{u}_j(x)\, L_j(\sigma t), \;\; x \in \Gamma,\, t \in \mathbb{R}_+, \tag{71}$$

where $u_j := (\mathcal{L}_j\mathcal{D}\lambda)$ and $\tilde{u}_j := (\mathcal{L}_j\mathcal{W}\lambda)$, are convergent in the corresponding Sobolev spaces.

Lemma 2. *Let* $\sigma > 0$ *be an arbitrary constant.*
(i) *If an arbitrary function* λ *belongs to space* $H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$*, then expansion (70) is convergent in the space* $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$*. If* $\lambda \in H^3_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$*, then expansions (70) and (71) are convergent in spaces* $L^2_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ *and* $L^2_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$*, correspondingly.*
(ii) *Coefficients* u_j, $\tilde{u}_j$, $j \in \mathbb{N}_0$, *are components of q-convolutions (34) and*

$$\tilde{\mathbf{u}}(x) = \mathbf{W} \underset{H^{-1/2}(\Gamma)}{\circ} \boldsymbol{\lambda}, \;\; x \in \Gamma, \tag{72}$$

correspondingly, where $\boldsymbol{\lambda} = \mathcal{L}\lambda \in l^2(H^{1/2}(\Gamma))$.

Proof. The first statement of this lemma follows from the fact that by Lemma 1 the retarded double layer potential with a density that is an element of the space $H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, belongs to space $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$. If $\lambda \in H^3_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in L^2_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$, and $\mathcal{W}\lambda \in L^2_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$. Then by Theorem 4 the Laguerre transform can be applied to both the potential and its normal derivative, and expansions (70) and (71) with obtained coefficients are convergent in the appropriate spaces.

Let us consider the retarded potential (4) with density $\lambda \in H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$ at an arbitrary point $x \in \Omega$, and apply formula (63) to it as to an element of the space $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$:

$$\begin{aligned} u_j(x) :=& \mathcal{L}_j\, \mathcal{D}\lambda(x) = \\ =& \frac{\sigma}{4\pi}\int\limits_{\mathbb{R}_+} e^{-\sigma t}L_j(\sigma t)\int\limits_{\Gamma} \boldsymbol{\nu}(y)\cdot\nabla_y\left(\frac{\lambda(z,t-|x-y|)}{|x-y|}\right)\Bigg|_{z=y} d\Gamma_y dt, \\ & j \in \mathbb{N}_0. \end{aligned} \tag{73}$$

As points x and y do not coincide (i.e. partial derivatives in inner integral are bounded) and $||u_j||_{H^1(\Omega)} < +\infty$, then we can change the order of integration according to the Fubini theorem

$$u_j(x) = \frac{1}{4\pi} \int_{\Gamma} \partial_{\vec{\nu}(y)} \left(\frac{\sigma}{|x-y|} \int_{\mathbb{R}_+} \lambda(z, t - |x-y|) e^{-\sigma t} L_j(\sigma t) dt \right) \Bigg|_{z=y} d\Gamma_y, \quad x \in \Omega. \tag{74}$$

Note that in the obtained expression, the inner integral is expressing the j-th Fourier-Laguerre coefficient of "retarded" function λ. Therefore, according to Lemma 5 and formulas (68),(14) we can write the following:

$$\begin{aligned} u_j(x) =& \frac{1}{4\pi} \int_{\Gamma} \partial_{\vec{\nu}(y)} \left(\frac{e^{-\sigma|x-y|}}{|x-y|} \sum_{i=0}^{j} \zeta_{j-i}(x-y) \lambda_i(z) \right) \Bigg|_{z=y} d\Gamma_y = \\ =& \sum_{i=0}^{j} \int_{\Gamma} \lambda_i(y) \partial_{\vec{\nu}(y)} e_{j-i}(x-y) \, d\Gamma_y, \;\; j \in \mathbb{N}_0, \;\; x \in \Omega, \end{aligned} \tag{75}$$

where $\lambda_j := \mathcal{L}_j \lambda, \; j \in \mathbb{N}_0$.

For an arbitrary fixed point $x \in \Omega$ all components of sequence $\mathbf{e}(x - \cdot)$ are continuously differentiable functions on Γ. Since $\lambda_j \in H^{1/2}(\Gamma), \; j \in \mathbb{N}_0$, then for the Lipschitz surface Γ integrals in (75) can be interpreted as the inner product of elements in $L^2(\Gamma)$ and can be extended to the duality relation on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$:

$$u_j(x) = \sum_{i=0}^{j} \left\langle \partial_{\vec{\nu}(\cdot)} e_{j-i}(x - \cdot), \lambda_i(\cdot) \right\rangle_{\Gamma}, \; x \in \Omega, \; j \in \mathbb{N}_0. \tag{76}$$

So we received coefficients of the q-convolution (34).

If $\lambda \in H^3_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma))$ we have $||u_j||_{H^1(\Omega,\Delta)} < +\infty$ and, obviously, for any point $x \in \Omega$ previous considerations regarding functions in integrals in formulas (73)-(76) hold. Therefore the form of coefficients $u_j, \; j \in \mathbb{N}_0$, is the same. Besides, for these coefficients as elements of the space $H^1(\Omega, \Delta)$, we can define linear continuous operator of normal derivative [4, Lemma 3.2, Theorem 1]. Let us show that $\tilde{u}_j = \gamma_1 u_j, \; j \in \mathbb{N}_0$.

Consider an arbitrary point $x \in \Gamma$ and apply the Laguerre transform to $\mathcal{W}\lambda$:

$$\begin{aligned} \tilde{u}_j(x) := \mathcal{L}_j \, \mathcal{W}\lambda(x) = \frac{\sigma}{4\pi} \int_{\mathbb{R}_+} e^{-\sigma t} L_j(\sigma t) \times \\ \boldsymbol{\nu}(x) \cdot \lim_{x' \to x} \nabla_{x'} \int_{\Gamma} \boldsymbol{\nu}(y) \cdot \nabla_y \left(\frac{\lambda(z, t - |x' - y|)}{|x' - y|} \right) \Bigg|_{z=y} d\Gamma_y dt < +\infty. \end{aligned} \tag{77}$$

If we move differentiation by normal at the point x out of the integral over the time variable, we receive $\tilde{u}_j(x) = \gamma_1 u_j(x)$. □

Note that we do not move outer differentiation inside the integral over the boundary Γ in order to avoid a high order of the singularity in a kernel. The definition of normal derivative operator γ_1 in case if $\mathbf{u} \in (H^1(\Omega, \Delta))^\infty$ was presented in [20]. In applications when calculating the respective singular integrals it is possible to replace normal derivatives with corresponding derivatives in the tangent plane (See, e.g. [1, formula (2.16)]).

6. Finding a generalized solution of the problem (1)-(3)

Consider operator

$$\mathcal{G} : \mathcal{H}^1_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma)) \to \mathcal{H}^0_\alpha(\mathbb{R}_+; H^1(\Omega)),\ \alpha = \sigma_0/2, \tag{78}$$

which maps the boundary value g to the generalized solution $u = \mathcal{G}g$ of the problem (1)-(3) according to the proposition 1. Taking into account the obvious inclusion

$$H^1_\sigma(\mathbb{R}_+; H^1(\Omega)) \subset \big(H^1_\sigma(\mathbb{R}_+; L^2(\Omega)) \cap L^2_\sigma(\mathbb{R}_+; H^1(\Omega))\big),$$

let us define a restriction of the operator $\mathcal{G}$ on elements from weighted Sobolev spaces.

Lemma 3. *Let $g \in H^{m+2}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then for arbitrary values $\sigma \geq \sigma_0$ operator*

$$\mathcal{G} : H^{m+2}_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma)) \to H^m_\sigma(\mathbb{R}_+; H^1(\Omega)) \tag{79}$$

is bounded.

Proof. Let g be an arbitrary function from the space $H^{m+2}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$. Considering it as an element of the space $\mathcal{H}^{m+2}_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with $\alpha = \sigma_0/2$, we will have the solution $u = \mathcal{G}g$. Let us estimate it using the inequality (46):

$$\begin{aligned} ||u||^2_{\mathcal{H}^m_\alpha(\mathbb{R}_+;H^1(\Omega))} &= \frac{1}{2\pi} \int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m} ||\hat{u}(\cdot,\omega)||^2_{H^1(\Omega)} d\omega \leq \\ &\leq \frac{\tilde{C}_1^2}{2\pi} \int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m} |\omega|^2 ||\hat{g}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)} d\omega = \\ &= \tilde{C}_1^2 ||g||^2_{\mathcal{H}^{m+2}_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))} < \infty. \end{aligned} \tag{80}$$

Since $u \in \mathcal{H}^m_\alpha(\mathbb{R}_+; H^1(\Omega))$, we get $u \in H^m_\sigma(\mathbb{R}_+; H^1(\Omega))$. □

Similarly, it is possible to examine the dependence of TDBIE solution on the smoothness (7) of the function g.

Lemma 4. *Let $g \in H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then there exists a unique solution of TDBIE (7) in the space $H^m_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma))$, and it satisfies the following condition with an arbitrary $\sigma \geq \sigma_0$:*

$$||\lambda||_{H^m_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq C||g||_{H^{m+1}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{81}$$

where $C > 0$ is a constant.

Proof. According to the proposition 1 consider operator

$$\mathcal{V}^{-1} : \mathcal{H}^1_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}^0_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$$

with the value $\alpha = \sigma_0/2$, that maps arbitrary function g to a unique solution of TDBIE $\lambda = \mathcal{V}^{-1}g$. With respect to the inequality (47), we get the following estimate for density λ:

$$\begin{aligned} ||\lambda||^2_{\mathcal{H}^m_\alpha(\mathbb{R}_+;H^{1/2}(\Gamma))} &= \frac{1}{2\pi} \int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m} ||\hat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)} d\omega \leq \\ &\leq \frac{\tilde{C}_2^2}{2\pi} \int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m}|\omega|^2 ||\hat{g}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)} d\omega = \\ &= \tilde{C}_2^2 ||g||^2_{\mathcal{H}^{m+1}_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))} < \infty, \end{aligned} \tag{82}$$

and inequality (81) implies here. □

Thus, Lemmas 3 and 4 specify the conditions regarding the function g, that cause the required smoothness of both the retarded potential density and the generalized solution of the problem (1)-(3) in weighted Sobolev spaces.

Proof of Theorem 1. Let boundary data in the boundary condition (3) be defined with function $g \in H^{m+3}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then, based on proposition 1, there exists a unique generalized solution of the problem (1)-(3) as element of the space $H^1_{\sigma_0}(\mathbb{R}_+; L^2(\Omega)) \cap L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$. In addition, we can conclude according with Lemma 3 that with boundary data specified below this solution belongs to the space $H^{m+2}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega)) \subset H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$, and for arbitrary $\sigma \geq \sigma_0$ following inequality holds:

$$||u||_{H^{m+2}_\sigma(\mathbb{R}_+;H^1(\Omega))} \leq C||g||_{H^{m+3}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{83}$$

where $C > 0$ is a constant that does not depend on g. Obviously, in that case estimate (36) is correct.

Consider now the TDBIE (7), having $g \in H^{m+3}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$. Then by Lemma 4 its solution λ belongs to space $H^{m+2}_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$. Based on this, the Laguerre transform is applicable to density λ (by Theorem 4) and $\boldsymbol{\lambda} := \mathcal{L}\lambda \in l^2(H^{1/2}(\Gamma))$. Furthermore, with such density the potential $\mathcal{D}\lambda$ belongs to the space of solutions of the problem (1)-(3), because $\mathcal{D}\lambda \in H^{m+1}_\sigma(\mathbb{R}_+; H^1(\Omega))$ by Lemma 1.

If $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$, then, according to Lemma 4 the density λ has to be element of the space $H^{m+3}_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and, by Lemma 1, we have $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+; H^1(\Omega, \Delta))$ and $\mathcal{W}\lambda \in H^m_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$. This means (by Lemma 4) that beginning from $m = 0$ the expansions (70) and (71) are convergent in spaces $L^2_\sigma(\mathbb{R}_+; H^1(\Omega, \Delta))$ and $L^2_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$, correspondingly, and the coefficients of these expansions have form of (34) and (72), correspondingly.

Let us build a sequence $\mathbf{g} := \mathcal{L}g \in l^2(H^{-1/2}(\Gamma))$ and substitute the Fourier-Laguerre expansion of the boundary function g in the right hand side of TDBIE (7). If we substitute the expansion (71) in its left hand side, we can equated the expressions beside Laguerre polynomials with the same index. As a result, we get an infinite triangular system of BIEs (35). It is known [20], that this system has a unique solution $\boldsymbol{\lambda}$. □

Consequently, the proposed method enables us to find the generalized solution of the Neumann problem for the homogeneous wave equation with homogeneous initial conditions using the Fourier-Laguerre expansion of the retarded double layer potential. Note that this approach can be adapted for finding the Cauchy datum of generalized solution using a Kirchhoff formula instead of retarded potential.

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ON THE BOUNDARY ELEMENT METHOD FOR BOUNDARY VALUE PROBLEMS FOR CONVOLUTIONAL SYSTEMS OF ELLIPTIC EQUATIONS

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Резюме. Для чисельного розв'язування крайових задач для нескінченних систем зі згортковою структурою, які складаються з еліптичних рівнянь другого порядку, запропоновано метод граничних елементів. Розв'язок подано за допомогою послідовності потенціалів простого шару. Для апроксимації невідомих густин потенціалів використано базис, який складається з кусково-сталих базисних функцій, побудованих на трикутних граничних елементах. Досліджено апріорні похибки. Наведено результати серії обчислювальних експериментів.

Abstract. For the numerical solution of boundary value problems for infinite systems with convolutional structure that consist of the second order elliptic equations, a boundary elements method is suggested. The solution is given as a sequence of single layer potentials. For the approximation of the unknown densities of the potentials a basis that consists of piece-wise constant functions built on triangular boundary elements is used. A priory error estimates are obtained. Results of a series of computational experiments are given.

1. Introduction

Boundary value problems for infinite systems that consist of elliptic partial differential equations (PDEs) can be found while investigating solutions of linear evolution problems for instance in the following works [3, 6, 10, 15, 16, 21]. Note that in [14] the well-posedness of such problems has been proven by transitioning to the corresponding variational formulations. Integral representations of the solutions of these boundary value problems that lead to equivalent boundary integral equations (BIEs) have been obtained. Properties of the BIEs method for exterior problems have been studied by the author in [17].

The main goal of the current article is such transformation of the obtained system of BIE that allows to efficiently apply the Bubnov-Galerkin method to it and prove its convergence. We also develop an algorithm for its solution by the boundary elements method (BEM) and investigate the approximation error of the obtained solution.

The paper is organized as follows. In Section 2 we formulate a Dirichlet BVP for an infinite triangular system of elliptic PDEs. We consider this problem in appropriate Sobolev spaces and introduce a notion of sequences and a new operation on them – q-convolution. In this section we also give an integral

Key words. Boundary value problems; boundary integral equations; elliptic equation; infinite system; boundary element method; convolutional system.

representation of the solution of the BVP by a combination of some surface potentials which reduces the problem to a system of BIEs.

In Section 3 we transform the system of BIEs into such sequence of BIEs all equations of which have the same boundary integral operator in the left hand side. It allows us to justify the application of the Bubnov-Galerkin method for finding the unknown functions – densities of the potentials. Afterwards, the main properties of the BEM and a priory error estimate of the numerical solution are obtained. In Section 4 some computational aspects of the systems of linear equations that appear as a result of the discretization of the BIEs are considered. Results of a series of computational experiments for the numerical solution of some model problems are given in Section 5. In this section an example of the application of the suggested approach for the solution of an initial-boundary value problem for the wave equation with homogeneous initial conditions is given. In the last section conclusions about the introduced method are given.

2. Formulation of the convolutional systems of PDE and BIE

Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected domain with a Lipschitz boundary Γ and $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$ be an exterior domain. We consider an infinite system in Ω^+

$$\begin{cases} c_0 u_0 - \Delta u_0 = 0, \\ c_1 u_0 + c_0 u_1 - \Delta u_1 = 0, \\ c_2 u_0 + c_1 u_1 + c_0 u_2 - \Delta u_2 = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ c_k u_0 + c_{k-1} u_1 + ... + c_0 u_k - \Delta u_k = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{1}$$

where $u_0, u_1, \ ..., \ u_k, \ ...$ are unknown functions, $c_0, c_1, \ ..., \ c_k, \ ...$ are some given constants and $c_0 > 0$. We investigate BVPs for system (1) that consist in finding its solutions that satisfy the Dirichlet condition on the boundary Γ

$$u_k|_\Gamma = \tilde{g}_k, \ k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{2}$$

where $\tilde{g}_i$ $(i \in \mathbb{N}_0)$ are given functions on Γ. In other words, we will consider the Dirichlet problem (1), (2).

Let X be an arbitrary linear space over the field of real numbers, $\mathbb{Z}$ – the set of integers. By X^∞ we denote a linear space of mappings $\mathbf{u} : \mathbb{Z} \to X$ satisfying $u(k) = 0$ when $k < 0$. For any element $\mathbf{u} \in X^\infty$ we have $u_k \equiv (\mathbf{u})_k := \mathbf{u}(k), k \in \mathbb{Z}$, and will write it as $\mathbf{u} := (u_0, u_1, ..., u_k, ...)^\top$. Henceforth we will call elements of X^∞ sequences.

Let $\widetilde{\mathbf{E}}(x, y) = \left(\widetilde{E}_0(x, y), \ \widetilde{E}_1(x, y), \ ...\right)^\top, \ x, y \in \mathbb{R}^3$, be a fundamental solution of the system (1) and sequence $\mathbf{E}(x, y) = (E_0(x, y), \ E_1(x, y), \ ...)^\top$ is calculated by the formula

$$E_i(x, y) := \widetilde{E}_i(x, y) - \widetilde{E}_{i-1}(x, y), \ i \in \mathbb{N}, \ \ E_0(x, y) = \widetilde{E}_0(x, y), \ x, y \in \mathbb{R}^3. \tag{3}$$

Note that $E_0(x,y)=\frac{e^{-\sqrt{c_0}|x-y|}}{4\pi|x-y|}$. As for the other components see, for example, [17].

Consider a sequence of functions $\mathbf{V}\xi(x)=(V_0\xi(x),\ V_1\xi(x),\ ...)^\top$ with components

$$V_j\xi(x):=\big(V_j\xi\big)(x)=\int\limits_{\Gamma}\xi(y)E_j(x,y)d\Gamma_y,\ j\in\mathbb{N}_0,\ x\in\mathbb{R}^3, \tag{4}$$

where ξ is a square integrable on Γ function. It is known [17] that sequence $\mathbf{u}(x)=(u_0(x),\ u_1(x),\ ...)^\top$ built for an arbitrary sequence $\boldsymbol{\mu}=(\mu_0,\ \mu_1,\ ...)^\top$ of square integrable on Γ functions by the rule

$$u_i(x)=\sum_{j=0}^{i}V_j\mu_{i-j}(x),\ i\in\mathbb{N}_0,\ x\in\mathbb{R}^3, \tag{5}$$

will satisfy the system (1). Then in order for the sequence $\mathbf{u}$ to be a solution of the Dirichlet problem for the given sequence $\mathbf{g}=(g_0,\ g_1,\ ...)^\top$ it is enough to find such sequence $\boldsymbol{\mu}$ that would satisfy on Γ the following equalities

$$\begin{cases} V_0\mu_0=g_0,\\ V_1\mu_0+V_0\mu_1=g_1,\\ V_2\mu_0+V_1\mu_1+V_0\mu_2=g_2,\\ \quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\ V_k\mu_0+V_{k-1}\mu_1+...+V_0\mu_k=g_k,\\ \quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \end{cases} \tag{6}$$

Lets introduce some notations. We will use the Lebesgue space $L_2(\Omega^+)$ and Sobolev spaces $H^1(\Omega^+)$ of real-valued scalar functions. Let $\gamma_0^+ : H^1(\Omega^+)\to H^{1/2}(\Gamma)$ be the trace operator, $H^{-1/2}(\Gamma):=\big(H^{-1/2}(\Gamma)\big)'$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Definition 4. Let $\mathbf{g}\in(H^{1/2}(\Gamma))^\infty$. Sequence $\mathbf{u}\in(H^1(\Omega^+))^\infty$ is called a generalized solution of the Dirichlet problem if it satisfies the system (1) in the sense of distributions and the boundary condition (2) in the sense of traces.

Definition 5 ([10]). Let X, Y and Z be arbitrary linear spaces and $q : X\times Y\to Z$ – some mapping. By a q-convolution of sequences $\mathbf{u}\in X^\infty$ and $\mathbf{v}\in Y^\infty$ we understand a sequence $\mathbf{w}\in Z^\infty$ whose components are defined by the following rule

$$w_i:=\sum_{j=0}^{i}q\left(u_{i-j},v_j\right),\ i\in\mathbb{N}_0, \tag{7}$$

and denote it $\mathbf{w}=\mathbf{u}\underset{q}{\circ}\mathbf{v}$.

In case when $X=H^{-1/2}(\Gamma),\ Y=H^{1/2}(\Gamma),\ Z=\mathbb{R}$ and $q(u,v):=<u,v>_\Gamma$, $u\in H^{-1/2}(\Gamma),\ v\in H^{1/2}(\Gamma)$, for the components of the q-convolution of arbitrary sequences $\mathbf{u}\in\big(H^{-1/2}(\Gamma)\big)^\infty$ and $\mathbf{v}\in\big(H^{1/2}(\Gamma)\big)^\infty$ we have the following formula

$$w_j = \sum_{i=0}^{j} < u_{j-i}, v_i >_\Gamma, \quad j \in \mathbb{N}_0, \tag{8}$$

and write $\mathbf{w} := \mathbf{u} \underset{\Gamma}{\circ} \mathbf{v}$.

Another example of q-convolution is related to linear operators, when $X = \mathcal{L}(Y, Z)$ is a space of linear operators that act from Y into Z, and $q(A, v) := Av$, $A \in \mathcal{L}(Y, Z)$, $v \in Y$. In this case for the components of the q-convolution of arbitrary sequences $\mathbf{A} \in \big(\mathcal{L}(Y, Z)\big)^\infty$ and $\mathbf{v} \in Y^\infty$ we obtain the formula

$$w_j = \sum_{i=0}^{j} A_{j-i} v_i, \quad j \in \mathbb{N}_0, \tag{9}$$

and write $\mathbf{w} := \mathbf{A} \underset{Z}{\circ} \mathbf{v}$.

Definition 6 ([14]). Let $\mathbf{V} : \big(H^{-1/2}(\Gamma)\big)^\infty \to \big(H^{1/2}(\Gamma)\big)^\infty$ be a sequence of operators that act by the rule (4), where we consider the inner product in $L^2(\Gamma)$ extended to the duality on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\boldsymbol{\mu} \in \big(H^{-1/2}(\Gamma)\big)^\infty$. Sequence

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu}(x) := (\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu})(x), \quad x \in \mathbb{R}^3, \tag{10}$$

is called a single layer potential of the system (1) on the surface Γ.

Using the introduced notations, we can rewrite the system (6) as

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu} = \mathbf{g} \text{ on } \Gamma. \tag{11}$$

We will call systems of type (11) that can be represented by a q-convolution systems with a convolutional structure. It is easy to see that the system of PDEs (1) also has a convolutional structure since the expressions in it's left had side (that are not related to the Laplacian) are components of the q-convolution of sequences $\mathbf{c}$ and $\mathbf{u}$.

Proposition 6 ([14]). *For an arbitrary sequence* $\mathbf{g} \in l^2(H^{1/2}(\Gamma))$ *there exists a unique generalized solution of the Dirichlet problem* $\mathbf{u} \in l^2(H^1(\Omega))$. *It can be represented as a single layer potential* (10) *whose density* $\boldsymbol{\mu} \in l^2(H^{-1/2}(\Gamma))$ *is a solution of the BIE* (11).

3. Boundary Elements Method for BIE System

Triangular shape of system (11) is a consequence of the convolutional structure of (1) and the application of the q-convolution in the single layer potential definition. Lets use this property to build a step-by-step process of the numerical solution of the BIE (11). This system can be represented as a sequence of Fredholm BIEs of the first kind:

$$V_0 \mu_k = \tilde{g}_k \quad \text{в} \quad H^{1/2}(\Gamma), \quad k \in \mathbb{N}_0, \tag{12}$$

where

$$\tilde{g}_k := g_k - \sum_{i=0}^{k-1} V_{k-i} \mu_i. \tag{13}$$

As you can see, the system is reduced to a sequence of equations that have the form

$$V_0\eta = f \quad \text{в} \quad H^{1/2}(\Gamma). \tag{14}$$

They have two important properties. First, the left-hand side of the integral equation with an arbitrary index $k \in \mathbb{N}$ is defined by the same boundary operator V_0 and the right-hand side depends on the boundary condition data and on the solutions of the equations with previous indexes $i = \overline{0, k-1}$. Taking these considerations into account during the the implementation of the method makes it possible to build efficient algorithms for the numerical solution of the obtained sequence of BIEs (12) as well as for the computation of the solutions of the boundary problem.

Another feature of the obtained system is that the boundary integral operator on left-hand side of the equations corresponds to the elliptic operator $c_0 I - \Delta$, where I is the identity operator, and is well studied in the literature (see, e.g., [2, 4, 5, 13]). In our case, it gives us the opportunity not only to prove the existence and the uniqueness of the solutions of the obtained sequence of BIEs, but also to get the corresponding numerical solutions using BEM, which is considered as a representative of the Bubnov-Galerkin method family [8]. A large number of publications (see, e.g., the literature review in [9, 20]) confirms the effectiveness and the versatility of this method regarding the numerical solution of boundary value problems for different types of elliptic equations and systems of elliptic equations of smaller dimension.

Investigation of the solutions of BIE (14) and the approximation by the Bubnov-Galerkin scheme is based on the ellipticity and the boundedness of the operator V_0:

$$\langle V_0\eta, \eta\rangle_\Gamma \geq c_1 ||\eta||^2_{H^{-1/2}(\Gamma))}, \quad ||V_0\eta||_{H^{1/2}(\Gamma))} \leq c_2 ||\eta||_{H^{-1/2}(\Gamma))}, \quad \forall \eta \in H^{-1/2}(\Gamma)),$$

where $c_1 > 0$ and $c_2 > 0$ are constants.

Consider a sequence of finite-dimensional subspaces $X_M \subset H^{-1/2}(\Gamma)$, $M \in \mathbb{N}$, that are linear spans of functions $\{\phi_i\}_{i=1}^M$ that form a basis in X_M. According to the Bubnov-Galerkin method, we seek a numerical solution of the equation (14) in the form of a linear combination

$$\eta^M := \sum_{i=1}^{M} \eta_i \phi_i \in X_M \tag{15}$$

as a solution of such variational problem

$$\langle V_0\eta^M, \eta\rangle_\Gamma = \langle f, \eta\rangle_\Gamma, \quad \forall \eta \in X_M. \tag{16}$$

In order to find the vector of the unknown coefficients $\boldsymbol{\eta}^{[M]} := \{\eta_i\}_{i=1}^M \in \mathbb{R}^M$ lets take the basis functions ϕ_j as the test ones. Then from the variational equations we obtain a system of linear algebraic equations (SLAE) regarding the unknown coefficients η_i:

$$V_0^{[M]} \boldsymbol{\eta}^{[M]} = \mathbf{f}^{[M]}, \tag{17}$$

where $V_0^{[M]}[j, i] := \langle V_0\phi_i, \phi_j\rangle_\Gamma$, $f_j^{[M]} := \langle f, \phi_j\rangle_\Gamma$, $\; i, j = \overline{1, M}$.

Note that the matrix of the obtained system if symmetric. Moreover, as a result of the $H^{-1/2}(\Gamma)$-ellipticity of the operator V_0, it is positive definite. Therefore, with an arbitrary right-hand side the system (17) will have a unique solution i.e. $\forall M \in \mathbb{N}$ by using the Bubnov-Galerkin method we will get an approximate solution of the equation (14). By the Cea lemma (see, e.g., [20, Theorem 8.1]) such approximate solution satisfies the inequality

$$||\eta_M||_{H^{-1/2}(\Gamma)} \leq c_1||f||_{H^{1/2}(\Gamma))}, \tag{18}$$

and there exists an estimate for its error

$$||\eta - \eta_M||_{H^{-1/2}(\Gamma)} \leq \frac{c_2}{c_1} \inf_{\xi \in X^M} ||\eta - \xi||_{H^{-1/2}(\Gamma)}. \tag{19}$$

Hence the convergence in $H^{-1/2}(\Gamma)$ of the approximate solution $\eta_M \to \eta \in H^{-1/2}(\Gamma)$ when $M \to \infty$, where η is the solution of the corresponding BIE in the sequence (12). Note that convergence of the numerical solution follows from the approximation property of the trial space X_M.

Lets specificate the numerical scheme (17) using the boundary elements method [8, 19, 20]. Let $\Gamma_{\widetilde{M}} = \bigcup_{l=1}^{\widetilde{M}} \overline{\tau}_l$ be some approximation of the surface Γ built by triangular boundary elements $\{\tau_l\}_{l=1}^{\widetilde{M}}$ with vertices $\{x^{[l_1]}, x^{[l_2]}, x^{[l_3]}\}$ and $h := \max\limits_{l=\overline{1,\widetilde{M}}} \left(\int_{\tau_l} \mathrm{d}s \right)^{1/2}$ – parameter of the approximation. We assume that vertices of all triangles have global numeration $\{x_k\}_{k=1}^{M^*}$.

Lets build a set of linearly-independent on $\Gamma_{\widetilde{M}}$ piece-wise constant functions $\{\varphi_l^0\}_{l=1}^{M}$, $M = \widetilde{M}$:

$$\varphi_l^0(x) = \begin{cases} 1, & x \in \tau_l, \\ 0, & x \notin \tau_l. \end{cases} \tag{20}$$

We will consider finite-dimensional spaces of functions $S_h^0(\Gamma) := X^M = \operatorname{span}\{\varphi_l^s\}_{l=1}^{M}$, $\dim S_h^0(\Gamma) = M$ as approximating spaces for the numerical scheme (17).

Let the operator equation (14) correspond to some k-th equation of the sequence (12). Its approximate (numerical) solution μ_k^h can be represented as a linear combination of piece-wise constant functions:

$$\mu_k^h = \sum_{l=1}^{M} \mu_{k,l}^h \varphi_l^0 \ \in S_h^0(\Gamma), \ \ k \in \mathbb{N}_0. \tag{21}$$

Here $\left\{\mu_{k,l}^h\right\}_{l=1}^{M} =: \boldsymbol{\mu}_k^h \in \mathbb{R}^M$ is a vector of unknown coefficients that can be found from the following system of algebraic equations:

$$\mathbf{V}_0^h \boldsymbol{\mu}_k^h = \tilde{\mathbf{g}}_k^h, \ \ k \in \mathbb{N}_0. \tag{22}$$

Matrix $\mathbf{V}_0^h$ is a concrete representation of the matrix of the system (17). Its elements can be given as

$$V_0^h[i,l] = \int_{\tau_i} \int_{\tau_l} E_0(x-y) \mathrm{d}s_y \mathrm{d}s_x, \ \ i,l = \overline{1,M}, \tag{23}$$

and the components of the right-hand side vector in (22) have the following form

$$\tilde{g}_k^h[i] = \int_{\tau_i} \Big\{ g_k(x) - \sum_{j=0}^{k-1} \big(V_{k-j}\mu_j^h\big)(x) \Big\} \mathrm{d}s_x, \;\; j = \overline{1, M}. \tag{24}$$

Sequence $\boldsymbol{\mu}^h := \big(\mu_0^h, \mu_1^h, ...\big)^\top$ can be treated as a numerical solution of the system of BIEs (12). After finding the consequent solution $\boldsymbol{\mu}_k^h$ of the algebraic system (22), we can approximate the corresponding density element using the formula (21) and calculate the k-th component of the numerical solution of the Dirichlet problem at an arbitrary point $x \in \Omega^+$:

$$u_k^h(x) = \sum_{j=0}^{k} \big(V_{k-j}\mu_j^h\big)(x), \; x \in \Omega^+. \tag{25}$$

The sequence $\mathbf{u}^h := \big(u_0^h, u_1^h, ...\big)^\top$ can be treated as a numerical solution of the Dirichlet problem.

Lets find an apriory estimate for the error of its components after introducing some Sobolev spaces [9]. Let the boundary Γ be given as a union $\Gamma = \bigcup_{i=1}^{\widetilde{N}} \overline{\Gamma}_i$ of surfaces Γ_i ($\Gamma_i \cap \Gamma_j = \oslash$ when $i \neq j$) each of which has a sufficiently smooth parameterization

$$\Gamma_i := \big\{ x \in \mathbb{R}^3 \, : \, x = \widetilde{\chi}_i(\xi), \; \xi \in \widetilde{\tau}_i \subset \mathbb{R}^2 \big\}.$$

By using a set of non-negative functions $\phi_i \in C_0^\infty(\mathbb{R}^3)$ such that

$$\sum_{i=1}^{\widetilde{N}} \phi_i(x) = 1 \;\; \forall x \in \Gamma, \quad \phi_i(x) = 0 \;\; \forall x \in \Gamma \setminus \Gamma_i,$$

each function v given on the boundary Γ can be written in a form

$$v(x) = \sum_{i=1}^{\widetilde{N}} \phi_i(x) v(x) = \sum_{i=1}^{\widetilde{N}} v_i(x) \;\; \forall x \in \Gamma, \tag{26}$$

where $v_i(x) := \phi_i(x)v(x) \;\; \forall x \in \Gamma_i$. We consider the Sobolev spaces $H^m(\widetilde{\tau}_i)$ when $m \in \mathbb{N}_0$, elements of which are functions $\widetilde{v}_i(\xi) := v_i(\widetilde{\chi}_i(\xi))$ when $\xi \in \widetilde{\tau}_i$, with a norm and a half-norm

$$||\widetilde{v}_i||_{H^m(\widetilde{\tau}_i)} := \Bigg(\sum_{|\alpha| \le m} ||\partial^\alpha \widetilde{v}_i||^2_{L^2(\widetilde{\tau}_i)} \Bigg)^{1/2}, \;\; |\widetilde{v}_i|_{H^m(\widetilde{\tau}_i)} := \Bigg(\sum_{|\alpha| = m} |\partial^\alpha \widetilde{v}_i|^2_{L^2(\widetilde{\tau}_i)} \Bigg)^{1/2}, \tag{27}$$

correspondingly. Here ∂^α is a notation of the partial derivative with a multi-index $\alpha = (\alpha_1, \alpha_2)$. Then for the functions, given on the whole surface Γ, we will use the Sobolev spaces $H^m(\Gamma)$ with a norm and a half-norm

$$||v||_{H^m(\Gamma)} := \Bigg(\sum_{i=1}^{\widetilde{N}} ||\widetilde{v}_i||^2_{H^m(\widetilde{\tau}_i)} \Bigg)^{1/2}, \;\; |v|_{H^m(\Gamma)} := \Bigg(\sum_{i=1}^{\widetilde{N}} |\widetilde{v}_i|^2_{H^m(\widetilde{\tau}_i)} \Bigg)^{1/2}, \tag{28}$$

correspondingly.

For non-integer values of the indexes $s = m + \sigma$, $m \in \mathbb{N}_0$, $\sigma \in (0,1)$, we will use Sobolev-Slobodetski spaces $H^s(\widetilde{\tau}_i)$ and $H^s(\Gamma)$ with corresponding half-norms and norms

$$|\widetilde{v}_i|_{H^s(\widetilde{\tau}_i)} := \left(\sum_{|\alpha|=m} \int_{\widetilde{\tau}_i} \int_{\widetilde{\tau}_i} \frac{|\partial^\alpha \widetilde{v}_i(\xi) - \partial^\alpha \widetilde{v}_i(\eta)|^2}{|\xi - \eta|^{2+2\sigma}} ds_\xi ds_\eta \right)^{1/2},$$

$$||\widetilde{v}_i||_{H^s(\widetilde{\tau}_i)} := \left(||\widetilde{v}_i||^2_{H^m(\widetilde{\tau}_i)} + |\widetilde{v}_i|^2_{H^s(\widetilde{\tau}_i)} \right)^{1/2}, \tag{29}$$

$$|v|_{H^s(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} |\widetilde{v}_i|^2_{H^s(\widetilde{\tau}_i)} \right)^{1/2}, \quad ||v||_{H^s(\Gamma)} := \left(||v||^2_{H^m(\Gamma)} + |v|^2_{H^s(\Gamma)} \right)^{1/2},$$

and also spaces of piece-wise smooth functions

$$H^s_{pw}(\Gamma) := \left\{ v \in L^2(\Gamma) : v|_{\Gamma_i} \in H^s(\Gamma_i) \right\}, \tag{30}$$

for which

$$||v||_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} ||v_{|\Gamma_i}||^2_{H^s(\Gamma_i)} \right)^{1/2}, \quad |v|_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} |v_{|\Gamma_i}|^2_{H^s(\Gamma_i)} \right)^{1/2}. \tag{31}$$

Lemma 1. *Let $\boldsymbol{\mu} \in \left(H^s_{pw}(\Gamma)\right)^\infty$ be a solution of the system* (12) *for some $s \in (0,1]$, that satisfies the inequality*

$$\sum_{j=0}^{\infty} |\mu_j|_{H^s_{pw}(\Gamma)} < +\infty. \tag{32}$$

Then for the components of the numerical solutions of the system of BIEs (12) *and the Dirichlet problem* (1), (2) *obtained by BEM the following asymptotic estimates hold*

$$\left\| \mu_k - \mu_k^h \right\|_{H^{-1/2}(\Gamma)} \le c_k h^{s+1/2} |\mu_k|_{H^s_{pw}(\Gamma)}, \; k \in \mathbb{N}_0, \tag{33}$$

$$|u_k(x) - u_k^h(x)| \le \tilde{c}_k h^{s+1/2} \sum_{j=0}^{k} |\mu_j|_{H^s_{pw}(\Gamma)}, \quad x \in \Omega^+, \quad k \in \mathbb{N}_0, \tag{34}$$

where c_k and $\tilde{c}_k$ are some values that do not depend on the parameter h.

Proof. Validity of the statement regarding (33) directly follows from a known theorem ([7], [20, Theorem 12.3]).

A priory error of the k-th component of the numerical solution of the Dirichlet problem at an arbitrary point $x \in \Omega^+$ can be given as

$$|u_k(x) - u_k^h(x)| = \Big| \sum_{i=0}^{k} V_{k-i}\big(\mu_i - \mu_i^h\big)(x) \Big| = \Big| \sum_{i=0}^{k} \langle \big(\mu_i - \mu_i^h\big), \; E_{k-i}(x - \cdot) \,\rangle_\Gamma \Big|.$$

Note, that for an arbitrary fixed point $x \in \Omega^+$ all the functions $E_j(x - \cdot)$ are infinitely-differentiable and bounded together with all their derivatives on Γ, i.e. $\left\|E_j(x-\cdot)\right\|_{H^{1/2}(\Gamma)} \leq c_j^* = const$. Using the generalized Cauchy-Schwarz inequality, we get

$$|u_k(x) - u_k^h(x)| \leq \sum_{i=0}^{k} \left|\langle(\mu_i - \mu_i^h),\ E_{k-i}(x-\cdot)\ \rangle_\Gamma\right| \leq$$
$$\leq \sum_{i=0}^{k} \left\|\mu_i - \mu_i^h\right\|_{H^{-1/2}(\Gamma)} \left\|E_{k-i}(x-\cdot)\right\|_{H^{1/2}(\Gamma)}.$$

Then, taking into account the inequality (33), we obtain

$$|u_k(x) - u_k^h(x)| \leq h^{s+1/2} \sum_{i=0}^{k} c_{k-i}^* c_i |\mu_i|_{H^s_{pw}(\Gamma)} \leq \tilde{c}_k h^{s+1/2} \sum_{i=0}^{k} |\mu_i|_{H^s_{pw}(\Gamma)},$$

where $\tilde{c}_k = \max\limits_{0\leq i\leq k} \{c_{k-i}^* c_i\}$ does not depend on the parameter h. □

4. Computational aspects of the method

Effectiveness of the numerical solution of the Dirichlet problem depends in great length on the approaches for the calculation of the surface potential in the domain and the trace on the boundary. In practice, it means a combination of algorithms for numerical integration and analytic calculation of some singular integrals over the boundary elements.

If the point, at which the trace of the potentials mentioned above is calculated, is not located on the boundary element over which the integration is performed, then the kernels of these potentials are infinitely-differentiable functions on the corresponding boundary element. Hence, the calculation of the majority of the elements in corresponding SLAE and also the components of the numerical solution of the problem at the observational points can be performed using numerical integration and the Gauss quadrature in particular.

Lets consider the calculation of integrals over singular functions that can be obtained during the construction of the matrix of the SLAE and correspond to the boundary operator $\mathbf{V}_0$ (23):

$$V_0^h[k,l] = \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_l} \frac{e^{-\sqrt{c_0}|x-y|}}{|x-y|} ds_y ds_x, \quad k,l = \overline{1,M}. \tag{35}$$

If the boundary elements τ_k and τ_l coincide or are adjacent then the integrand of the internal integral has a weak singularity when the points $x \in \tau_k$ and $y \in \tau_l$ coincide. It can be explicitly eliminated if the element of the matrix is given as

$$V_0^h[k,l] = \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_l} \frac{e^{-\sqrt{c_0}|x-y|} - 1}{|x-y|} ds_y ds_x + \frac{1}{4\pi} \int_{\tau_k} I_l(x) ds_x, \tag{36}$$

where

$$I_l(x) = \int_{\tau_l} \frac{1}{|x-y|} ds_y. \tag{37}$$

Integrand of the first integral in (36) allows continuous definition at $x = y$ (it can be verified if the exponential function is expanded in a Maclaurin series over the variable $r = |x-y|$), so the value of this integral can be found numerically using the Gauss quadrature rules. The integral (37) can be found analytically as a function [11,18–20], parameterized by the geometric data of the boundary element τ_l and the coordinates of the point x.

In the integrals

$$V_j^h[k,l] = \int_{\tau_k}\int_{\tau_l} E_j(x,y)ds_y ds_x, \quad k,l=\overline{1,M}, \; j\in\mathbb{N}, \tag{38}$$

that correspond to the boundary operator $\mathbf{V}_j$, $j\in\mathbb{N}$, and are found during the construction of the right-hand side, the integrands are continuous for any location of the boundary elements τ_k and τ_l. Hence these integrals can also be found numerically using the Gauss quadratures.

Note, that all relations of the suggested approach can be applied to interior BVP without any changes.

5. Results of the computational experiment

Lets demonstrate the usage of the suggested method to find numerical solutions of some model Dirichlet problems. We assume that in (1) and (2) components of the sequences $\mathbf{c}$ and $\mathbf{g}$ have the form $c_k=(k+1)\kappa$ and $g_k=v_k$, $k\in\mathbb{N}_0$, correspondingly, where κ is some parameter and the sequence $\mathbf{v}$ consists of functions

$$\begin{aligned} v_k(x) =& \frac{e^{-\kappa(|x-x^*|-1)}\big(L_k(\kappa(|x-x^*|)-L_{k-1}(\kappa(|x-x^*|)\big)}{|x-x^*|}, \quad k\in\mathbb{N}, \\ v_0(x) =& \frac{e^{-\kappa(|x-x^*|-1)}}{|x-x^*|}, \end{aligned} \tag{39}$$

parameterized by some point x^*, L_k, $k\in\mathbb{N}_0$, are the Laguerre polynomials [1]. Up to a factor the sequence $\mathbf{v}$ coincides with the fundamental solution of the system (1), so it will be used to build the analytical solution of the Dirichlet problem. Note, that the variable x will denote points on the boundary Γ and in the domain where the numerical solution is sought, and the parameter x^* is located in the complement of this domain to the whole space $\mathbb{R}^3$.

We consider the following domains in the model problem: a unit sphere, its exterior in $\mathbb{R}^3$, a cube $\Omega := (-1,1)\times(-1,1)\times(-1,1)$ and its exterior $\Omega^+ := \mathbb{R}^3\setminus\overline{\Omega}$.

Lets consider first the model boundary value problems for the first equation of the system (1).

Example 1. Find a numerical solution u_0^h of the exterior ($x^*=(0,0,0)$) and interior ($x^*=(2,0,0)$) Dirichlet problems in case of the cubic boundary when $g_0=v_0$.

Table 1 contains corresponding numerical solutions of the exterior problem using the decomposition of the cube's boundary into $\overline{M}=1200$ boundary elements. As we can see, with increasing value of κ the solutions are decreasing rapidly when moving further from the boundary. Next, we examine the errors of the numerical solutions of this problem with a fixed value of the parameter κ, for example, take $\kappa=2$.

TABL. 1. Numerical solutions $u_0^h(x)$ of the problem 1 for different values of κ

x_1	Value of the parameter κ				
	0.5	1.0	2.0	4.0	8.0
1.2	$7.48314 \cdot 10^{-1}$	$6.75199 \cdot 10^{-1}$	$5.49666 \cdot 10^{-1}$	$3.64214 \cdot 10^{-1}$	$1.59908 \cdot 10^{-1}$
2.0	$3.02031 \cdot 10^{-1}$	$1.82901 \cdot 10^{-1}$	$6.70731 \cdot 10^{-2}$	$9.01743 \cdot 10^{-3}$	$1.62947 \cdot 10^{-4}$
3.0	$1.22230 \cdot 10^{-1}$	$4.49190 \cdot 10^{-2}$	$6.06765 \cdot 10^{-3}$	$1.10698 \cdot 10^{-4}$	$3.68426 \cdot 10^{-8}$
4.0	$5.56144 \cdot 10^{-2}$	$1.23988 \cdot 10^{-2}$	$6.16492 \cdot 10^{-4}$	$1.52428 \cdot 10^{-6}$	$9.29979 \cdot 10^{-12}$

TABL. 2. Errors of the numerical solution $u_0^h(x)$ of the problem 1

	Exterior problem			Interior problem		
$\overline{M}$	δ^h	eoc	$\epsilon^h(\%)$	δ^h	eoc	$\epsilon^h(\%)$
300	0.01384		3.10	0.01324		2.99
588	0.00702	2.018	1.55	0.00673	2.012	1.50
768	0.00537	2.010	1.18	0.00515	2.005	1.14
972	0.00421	2.061	0.93	0.00404	2.058	0.90
1200	0.00340	2.030	0.75	0.00326	2.027	0.72
1728	0.00234	2.039	0.51	0.00225	2.037	0.50
2700	0.00149	2.033	0.33	0.00143	2.031	0.32

In order to find the dependency between the error of the numerical solution and the parameter h that defines the triangulation of the boundary surface we will consider the values $\delta^h := ||u_0^h - u_0||_{L^2(a,b)}$ and $\epsilon^h := \frac{\delta^h}{||u_0||_{L^2(a,b)}} \cdot 100\%$, where (a, b) is an inverval in space from which the points of observation x are taken. We will also calculate the value of the estimated order of convergence [19]

$$eoc := \frac{\ln \delta^{h_j} - \ln \delta^{h_{j+1}}}{\ln h_j - \ln h_{j+1}}, \tag{40}$$

where h_j and h_{j+1} are the parameters of the two consequent triangulations of the boundary surface into boundary elements. Results of the calculations given in table 2 highlight the equal orders of errors of the numerical solutions of the interior and exterior problems. Moreover, the obtained result has $eoc \approx 2.0$.

Now lets demonstrate that the developed method gives us ability to find components of the numerical solutions with other values of the indexes.

Example 2. Find N components of the numerical solution u_i^h, $i = \overline{0, N}$, of the exterior Dirichlet problem (1), (2) if $\tilde{h}_i = v_i$, $\kappa = 2$ and $x^* = (0, 0, 0)$.

Charts of the obtained numerical solutions are given on figure 1. They demonstrate rapid decrease of the functions $u_i^h(x)$, $i = 0, 10, 20$, with the increase of their index. Numerical solutions obtained on $\overline{M} = 1200$ boundary elements are given in table 3 and indicate the commensurability of the errors of components of the numerical solutions $u_i^h(x)$ when $i = 10$ and $i = 20$ with the corresponding error of $u_0^h(x)$.

TABL. 3. Solutions $u_i^h(x)$, $i = 10, 20$ of the problem 2 when $\overline{M} = 1200$.

x_1	$u_{10}(x)$	$u_{10}^h(x)$	$u_{20}(x)$	$u_{20}^h(x)$
1.5	$8.8570 \cdot 10^{-2}$	$9.0932 \cdot 10^{-2}$	$-7.6672 \cdot 10^{-2}$	$-7.5956 \cdot 10^{-2}$
2.0	$5.6502 \cdot 10^{-2}$	$5.6254 \cdot 10^{-2}$	$4.0784 \cdot 10^{-2}$	$4.1496 \cdot 10^{-2}$
3.0	$-1.9676 \cdot 10^{-2}$	$-1.9664 \cdot 10^{-2}$	$-1.0549 \cdot 10^{-2}$	$-1.0619 \cdot 10^{-2}$
4.0	$4.3413 \cdot 10^{-3}$	$4.3359 \cdot 10^{-3}$	$2.9939 \cdot 10^{-3}$	$3.0045 \cdot 10^{-3}$

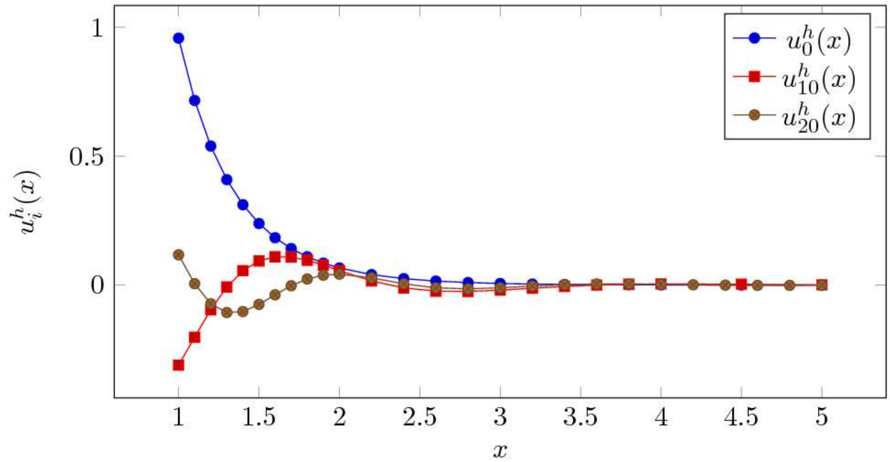

FIG. 1. Charts of the components $u_0^h(x)$, $u_{10}^h(x)$, $u_{20}^h(x)$ of the numerical solution of the problem 2 when $\overline{M} = 768$

As it has been mentioned above, the Dirichlet problem (1), (2) can be obtained by means of the application of the Laguerre transform by the time variable to a certain class of linear evolutionary problems. For instance, the system (1), that is mentioned in problems 1 and 2, can be obtained from a homogeneous wave equation with homogeneous boundary conditions. After finding for some N the components u_i^h, $i = \overline{0, N}$, the numerical solution of the mixed problem can be given as a partial sum of the Laguerre-Fourier expansion

$$u^{h,N}(x,t) = \frac{1}{\kappa} \sum_{i=0}^{N} u_i^h(x) L_i(\kappa t), \quad (x,t) \in \Omega^+ \times (0, \infty). \tag{41}$$

To generate the data for the boundary conditions (2) we use a "spherical impulse" with a center at x^*

$$v(x,t) = \frac{f(t - |x - x^*|)}{4\pi |x - x^*|}, \quad (x,t) \in \overline{\Omega^+} \times [0, \infty), \tag{42}$$

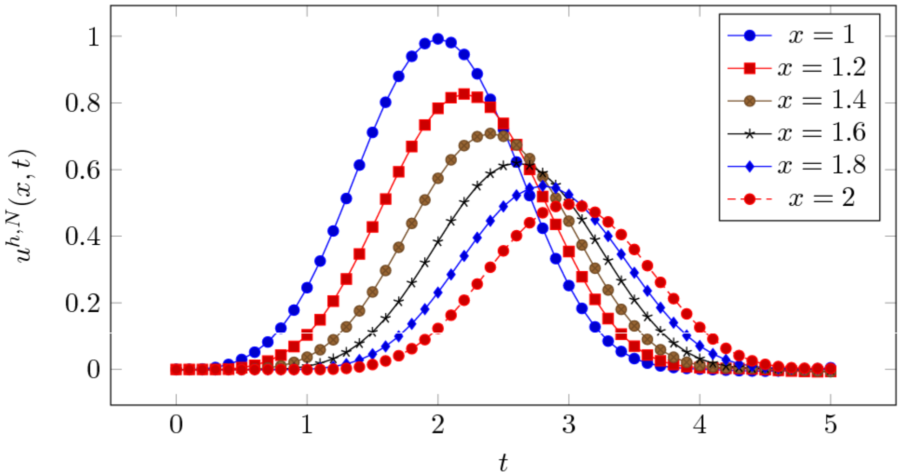

FIG. 2. Chart of the solution of the problem 3 in the exterior of the sphere with $N = 40$, $\overline{M} = 720$

where f is a cubical β-spline [12], and apply to it the Laguerre transform

$$v_k(x) = \int\limits_{\mathbb{R}_+} v(x,t)\, L_k(\kappa t)\, e^{-\kappa t} dt, \quad x \in \Gamma, \ k \in \mathbb{N}_0. \tag{43}$$

Example 3. In the exterior Ω^+ of the unit sphere calculate the numerical solution of the Dirichlet problem for the wave equation with homogeneous initial conditions and the boundary condition defined by (42) at $x^* = (0, 0, 0)$.

Let the problem (1), (2) correspond to the initial-boundary value problem 3 when $\kappa = 2$. After finding $N = 40$ components of the numerical solution u_i^h, $i = \overline{0, N}$, with the use of $\overline{M} = 720$ boundary elements, the numerical solution of the problem 3 at the points along the axis Ox_1 is calculated by the formula (41). As it can be seen from the charts of the numerical solution, given on the figure 2, the obtained results are well representing the physics of the wave propagation from the boundary surface, especially, passing through the observation points of the front and rear disturbance fronts.

Note that the formulation of the problem 3 gives us ability to find the coefficients u_i, $i \in \mathbb{N}_0$, of the expansion of the precise solution $u(x,t)$ into series (41) analytically. So it can be compared how the partial sums of the series (41) with analytical coefficients and coefficients found by the suggested approach approximate the precise solution of the evolution problem. As it can be see from the table 4, values of such partial sums are pointwise (regarding the time variable) close.

6. Conclusions

Application of the surface potentials built using the q-convolution operation is an effective way to obtain the integral representation of the solutions of

TABL. 4. Comparison of the numerical solution of the problem 3 $u^{h,N}(x,t)$ (the row above) with the values of the partial sum (41)(the row below), in which the coefficients are calculated analytically

t	$x_1 = 1.0$	$x_1 = 1.2$	$x_1 = 1.4$	$x_1 = 1.6$	$x_1 = 1.8$	$x_1 = 2.0$
0.0	0.00037	0.00043	-0.00011	0.00013	-0.00010	-0.00013
	0.00012	-0.00034	0.00012	0.00000	0.00008	-0.00004
0.4	0.01521	0.00136	-0.00007	-0.00006	-0.00003	-0.00002
	0.01595	0.00178	-0.00006	-0.00004	0.00004	0.00001
1.2	0.41588	0.20541	0.08926	0.03234	0.00852	0.00100
	0.42386	0.20880	0.09113	0.03376	0.00898	0.00100
2.0	0.99249	0.78406	0.57393	0.38364	0.23108	0.12327
	0.99860	0.78846	0.57774	0.38853	0.23510	0.12553

boundary value problems for infinite systems of PDE with convolutional structure. Such approach makes it possible to reduce the boundary value problem to an equivalent BIE system, develop efficient projection methods for its numerical solution and justify their usage. The results of a series of numerical experiments that confirm the theoretical statements and demonstrate the applicability of the proposed methods for modeling of evolutionary processes are given.

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ANOTHER CASE OF INCIDENCE MATRIX FOR BIVARIATE BIRKHOFF INTERPOLATION

A. Nazarzadeh, KH. Rahsepar Fard, A. Mahmoodi

Резюме. У цій статті спершу подано спеціальний випадок одновимірної задачі інтерполяції Біркгофа і за її допомогою апроксимовано розв'язок граничної задачі для рівняння Лапласа. Далі розглянуто інший тип двовимірної задачі інтерполяції Біркгофа, в якій умови інтерполяції задані в точках з кратністю. Введено інше позначення для матриці інцидентності. Зроблено порівняння апроксимацій Біркгофа і Хаара і показано перевагу інтерполяції Біркгофа.

Abstract. In this paper, first we present a special case of the univariate Birkhoff interpolation problem, and using that, we approximate the solution of a Laplace boundary value problem. Then we present another type of bivariate Birkhoff interpolation problem in which interpolation conditions are on some knots with multiplicity. We introduce another notation for incidence matrix. Finally, we compare two approximations Birkhoff and Haar then we show that Birkhoff interpolation is better than the other.

1. Introduction

In this paper we present some basic notations and useful properties in analyzing the interpolation polynomials. Let us denote Π_n the space of one variable interpolation polynomials of degree not exceeding n, and Π_n^2 the space of bivariate interpolation polynomials of degree not exceeding n.

The problem of interpolating a real function f by a univariate polynomial from the values of f and some of its derivatives on a set of knots is one of the main questions in numerical analysis and approximation theory.

In [1] and [10] the authors studied univariate Birkhoff interpolation and its properties. Let $x = \{x_1, ..., x_n\}$ be a set of real numbers such that $x_1 < ... < x_n$, let r be an integer and let $I \subset \{1, ..., n\} \times \{0, .., r\}$ be the set of pairs (i, j) in which the value $f^{(j)}(x_i) = f_{i,j}$ is known where f is a real function. The problem of determining the existence and uniqueness of a polynomial P in $\mathbb{R}^1$ satisfying the conditions $\forall (i, j) \in I, p^{(i)}(x_i) = f_{i,j}$ is called the *Birkhoff interpolation problem.*

In recent years there has been renewed interest and progress on Hermite-Birkhoff interpolation. The original source for this activity is work by G. D. Birkhoff in 1906, with a notable contribution by G. Polya in 1931.

Key words. Bivariate Birkhoff Interpolation Problem; Polya Condition; Incidence Matrix; Interpolation Polynomial; Haar Approximation; Hermite-Birkhoff; Operator Interpolation.

The interpolation conditions can be described by using special type matrices. Consider the matrix $E = (e_{i,j})$ with n rows and r+1 columns, filled with 0's and 1's so that $e_{i,j} = 1$ if and only if $(i,j) \in I$. The E is called *incidence matrix.*

In 1966 Schoenberg (see [19]) posed the problem of determining all those E for which the problem $P^{(j)}(x_i i) = c_{i,j}$ is always (for all choice of $x_i, c_{i,j}$) solvable. We call such matrices E *regular* and the remaining matrices *singular*.

Let $E = (e_{i,j})$ be an $m \times (n+1)$ incidence matrix. Then $m_j = \sum_i e_{i,j}$ is the number of 1's in column j, and $M_r = \sum_{j=0}^{r} m_j = \sum_{j=0}^{r}\sum_{i=1}^{m} e_{i,j}$ is the number of 1's in columns of E numbered $0, 1, \ldots, r$. For the matrix E, the condition $M_r \geq r+1, r = 0.1, ..., n$, is called the Polya condition.

Definition 7. The incidence matrix $E = (e_{i,j}), 1 \leq i \leq m, 0 \leq j \leq n$ is called poised with respect to $\{x_i\}_{i=1}^{m}$ if the unique solution of problem $P^{(j)}(x_i) = 0, 1 \leq i \leq m, 0 \leq j \leq n$ is a trivial polynomial.

In [8], the following Polya's result is well-known.

Theorem 1 (Polya's Theorem)**.** *The incidence matrix E of* $2 \times n$ *dimension is poised if and only if Polya condition is true.*

In [20], the author posed, for a $2 \times n$ incidence matrix $E = (e_{i,j})$, we define a $2 \times n$ matrix $G = (g_{i,j})$ as follows:

$$g_{i,j} = 1 - e_{i,n-j-1}, 1 \leq i \leq 2, 0 \leq j \leq n-1.$$

Then G is also an incidence matrix, because $\sum_{i=1}^{2}\sum_{j=0}^{n-1} e_{i,j} = n$. The matrix G is called a dual incidence matrix corresponding to E. For example, for the incidence matrix $E = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{Vmatrix}$, its dual matrix becomes $E' = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$.

The following theorem give a relationship between a $2 \times n$ incidence matrix E and its dual matrix G.

Theorem 2. *A* $2 \times n$ *incidence matrix E is poised if and only if dual matrix G is poised. In [20], the author shown that there exists a quadrature formula in the form* $\int_a^b P(x)dx = \sum_{e_{i,j}=1} w_{i,j} P^{(j)}(x_i)$ *to be exact for any polynomial P with degree at most n-1, where* $w_{i,j}$*'s are weight coefficients independent of P. This is called, the Hermite-Birkhoff quadrature formula for the incidence matrix E.*

Theorem 3. *A* $2 \times n$ *incidence matrix E is poised if and only if there exists a Hermite-Birkhoff quadrature formula specified by E.*

In [16], author presented below property of incidence matrix E:

Let $m_j = \sum_i e_{i,j}, j = 0, ..., n$ and $M_r = \sum_{j=0}^{r} m_j, r = 0, ..., n$, then E satisfies the strong Polya condition if $M_r \geq r+2, r = 0, ..., n-1$. If E does not satisfy strong Polya condition, then E may be decomposed in to matrices, $E = E_1 \bigoplus E_2 \bigoplus \cdots \bigoplus E_N$ where E_j's satisfies strong Polya condition.

In [18], the author proved below theorem:

Theorem 4. *Let* E_n *satisfy the Polya condition. Then* E_n *has a unique decomposition* $E_n = E_{n_1} \bigoplus E_{n_2} \bigoplus ... \bigoplus E_{n_q}, n_1 + n_2 ... + n_q = n$*, where* $E_{n_j}, j = 1, ..., q$

satisfies the strong Polya condition. Moreover E_n is poised if and only if E_{n_j}'s are poised.

The Birkhoff interpolation problem is one of the most general problems in multivariate interpolations. For clarity of the exposition, we will only restrict ourselves to the bivariate case.

In [11, Def. 3.1.1, p. 9], the authors studied bivariate Birkhoff interpolation problem. The bivariate Birkhoff interpolation problem depends on a finite set $T = \{z_q\}_{q=1}^m \subset \mathbb{R}^2$ of knots and interpolation space Π_n^2 of polynomials and an incidence matrix $E = (e_{q,\alpha})$. The bivariate Birkhoff interpolation problem is, for given real numbers $c_{q,\alpha}$, to find a polynomial $p \in \Pi_n^2$ satisfying the interpolation conditions

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial y^{\alpha_2}\partial x^{\alpha_1}} p(z_q) = c_{q,\alpha} \tag{1}$$

with $e_{q,\alpha} = 1$ where $\alpha = (\alpha_1, \alpha_2)$.

In this paper, we present a special case of univariate Birkhoff interpolation problem together with an example of boundary value problem introduce in [2], and also a method for obtaining the interpolation polynomial in the case of a set of types conditions, given on a set of knots in $\mathbb{R}^2$. This method is a generalization of the tensorial product method introduced by F.J.Hack in [7]. In this way, we investigate bivariate Birkhoff polynomial for the set of knots T such that $|T| < \binom{n+2}{2}$.

In [11] and [3], authors introduced Polya conditions for multivariate Birkhoff interpolation as follows:

Definition 8. An incidence matrix E satisfies the (lower) Polya condition (with respect to S) if $|E_A| \leq |A|$ for any lower set $A \subseteq S$. E satisfies the upper Polya condition if $|E_B| \leq |B|$ for any upper set $B \subseteq S$. A set B is an upper set with respect to S if $\alpha \in B, \beta \geq \alpha$ and $\beta \in S$ imply that $\beta \in B$. B is an upper set of S if and only if $S \setminus B$ is a lower set.

Similar to notations in [4], we apply the Haar function and interpolation problem. In [4], the authors presented some theorems for uniqueness. Thus we employ those theorems, for example, formula (8) and Theorem 3.1 and Example 3.2, p.107-109.

Problems of generalization in functions interpolation theory with functionals and operators in abstract spaces are considered in numerous works.

Definition 9. Let $F : X \longrightarrow Y$ be an operator, where X is a Hilbert and Y is a vector space; let $P_n : X \longrightarrow Y$ be an operator polynomial of the form $P_n(x) = L_0 + L_1 x + ... + L_n x^n$, where $L_0 \in Y$; and let $L_p(t_1, ..., t_p) : X^p \longrightarrow Y$ be a p-linear operator, $p = 1, ..., n$. Let $\{x_i\}_{i=1}^m$ be a system of elements from X. A polynomial operator P_n is called an interpolating polynomial for F in nodes $\{x_i\}_{i=1}^m \subset X$ if it satisfies the conditions $P_n(x_i) = F(x_i), i = 1, ..., m$.

In the case $X = Y = R^1$ the requirement that the interpolation functionals be the same algebraic polynomials.

In [12] and [13], the authors investigated the operator interpolation theory in Hilbert space and solvability Hermite interpolation problem with the operator values at the nodes with Gateaux differentials defined on the auxiliary nodes and some given directions. For example, let Π_n be a set of the operator polynomials $P_n : X \longrightarrow Y$ of degree not exceeding n and $p \in \Pi_n$ satisfies the conditions:

$$p(x_i) = F(x_i), p'(x_i)h_i = F'(x_i)h_i, i = 1, ..., m \tag{2}$$

For investigate Hermite problem with interpolation conditions (2) we consider the auxiliary nodes

$$\overline{x}_1 = x_1, \overline{x}_2 = x_1 + \alpha h_1, \overline{x}_3 = x_2, \overline{x}_4 = x_2 + \alpha h_2, ..., \overline{x}_{2m-1} = x_m,$$

$$\overline{x}_{2m} = x_m + \alpha h_m, \alpha \in R^1, \alpha \neq 0$$

of the matrix

$$\Gamma(\alpha) = \left\| \textstyle\sum_{p=0}^{n} (\overline{x}_i, \overline{x}_j)^p \right\|_{i,j=1}^{2m}$$

and

$$C(\alpha) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{-1}{\alpha} & \frac{1}{\alpha} & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\alpha} & \frac{1}{\alpha} & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & . & ... & . & . & . \\ . & . & . & . & . & ... & . & . & . \\ . & . & . & . & . & ... & . & . & . \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & \frac{-1}{\alpha} & \frac{1}{\alpha} \end{matrix} \right\|$$

and the vectors

$$F(\alpha) = (F(\overline{x}_1), F(\overline{x}_2), ..., F(\overline{x}_{2m})), P(\alpha) = (p(\overline{x}_1), p(\overline{x}_2), ..., p(\overline{x}_{2m})), p \in \Pi_n.$$

In [13], p.97, Theorem 1.1 shown that a necessary and sufficient condition for the solvability of the Hermite operator interpolation problem (2) in a Hilbert space, that the condition $ZF_H = 0$ and the formula $p(x) = q(x) + \langle F_H - q_H, H^+ g_H(x) \rangle$, with $q(x)$ varies over Π_n, describes the whole set of the Hermite operator polynomials of the n-th degree satisfies the interpolation conditions (2). In [13], explained notations $ZF_H, F_H, q_H, H^+ g_H$.

When some of the conditions of the Hermite interpolation are absent then, they are called to Hermite- Birkhoff conditions. For example, the conditions:

$$p(x_i) = F(x_i), p''(x_i)h_2^{(2)} h_1^{(2)} = F''(x_i)h_2^{(2)} h_1^{(2)}, i = 1, ..., m \tag{3}$$

are Hermite-Birkhoff conditions. In [13], Theorem 2.1, p.110, itroduced a necessary and sufficient condition for the solvability of the Hermite-Birkhoff operator interpolation problem in a Hilbert space.

Now we introduce an important result as follows:

A sufficient condition for the invariant solvability of the Hermite operator interpolation problem given as the following theorem. We shall denote by M a number of the interpolation conditions in the Hermite operator interpolation.

Theorem 5. *The Hermite interpolation problem in a Hilbert space is invariant solvability for any $n \geq M-1$.*

For example every Hermite interpolation problem with conditions (2) by Theorem 5 is invariant solvable.

By text in [13], p.112, if the Hermite-Birkhoff interpolation problem for a function of one variable has the unique solution, then the appropriate Hermite-Birkhoff operator interpolation problem is invariantly solvable. Now we apply Polya's theorem in case m=2 for the invariant solvability of the Hermite-Birkhoff operator problem with the interpolation conditions containing values of operator polynomial p of the third degree and Gateaux differentials of the second order

$$p(x_1), p''(x_1)h_2^{(1)}h_1^{(1)}, p(x_2), p''(x_2)h_2^{(2)}h_1^{(2)} \tag{4}$$

In the corresponding Hermite-Birkhoff interpolation problem of one variable we have

$$M=4, \quad n=M-1=3, \quad E=\left\|\begin{matrix}1 & 0 & 1\\ 1 & 0 & 1\end{matrix}\right\|,$$

$$m_0=2, \quad m_1=0, \quad m_2=2, \quad M_0=2, \quad M_1=2, \quad M_2=4$$

Since $M_j \geq j+1, j=0,1,2$ then by Polya's Theorem, the classical Hermite-Birkhoff problem

$$p(t_1)=0, \quad p''(t_1)=0, \quad p(t_2)=0, \quad p''(t_2)=0$$

on the set of the polynomial of the 3-d degree has the unique solution zero-polynomial. But as we stated above, the corresponding Hermite-Birkhoff operator problem (4) is invariantly solvable.

2. Bivariate Birkhoff Interpolation

Following R.A. Lorentz in [11], an interpolation problem is *regular* if it is uniquely solvable for all selections of distinct nodes and all data. In the univariate case, Lagrange and Hermite interpolation are regular, but in the multivariate case, even Lagrange interpolation is not regular. Here, we study a solvable interpolation problem in multivariate case.

A uniqueness technique for bivariate Birkhoff interpolation problem is presented in [7]. The technique has been explained in [7, theorem 3.3, p.26], where interpolation polynomial is tensor product of functionals. that is why, we introduce incidence matrix. For exactly M+1 pairs $(i,k) \in \{1,...,m\} \times \{0,...,M\}$, we suppose that $E_{i,k} = (e_{i,j}^{k,l})_{1 \leq j \leq a_{i,k}, 0 \leq l \leq N_{i,k}}$ where $a_{i,k} \in N, N_{i,k} \in N_0$ and for others $(i,k)'s, E_{i,k} = 0$. Regularity condition is established, using bidimensional incidence matrix corresponding to Birkhoff interpolation problem. Hence, the bivariate Birkhoff interpolation problem is as follows:

$$C^M(G), \sum_{s=1}^{p} \Pi_{M_s} \otimes \Pi_{N_s}; D_{x_i, y_{i,k,j}}^{k,l} : (i,k) \in Z, (x_i, y_{i,k,j}) \in T \tag{5}$$

This means that for all $f \in C^M(G), G \subset \mathbb{R}^2$ there exists $P \in \sum_{s=1}^{p} \Pi_{M_s} \otimes \Pi_{N_s}$ where $\Pi_{M_s} \otimes \Pi_{N_s}$ is tensor product of functionals and $D_{x_i,y_{i,k,j}}^{k,l} P = D_{x_i,y_{i,k,j}}^{k,l} f$ such that T is the set of distinct knots i.e. $T = \{(x_i, y_{i,k,j})\}$ s.t. $x_1 < ... < x_m$

and also $y_{i,k,1} < ... < y_{i,k,a_{i,k}}, (i,k) \in Z, Z \subset \{1,...,m\} \times \{0,...,M\}$ so that $Z = \{(i,k) : E_{i,k} \neq 0\}$.

In view of corollary 3.4 [7, p.27], if matrices E_s's for points $x_1,...,x_m$ are regular and matrices $E_{i,k}$'s are regular for points $y_{i,k,1},...,y_{i,k,a_{i,k}}$ then the incidence matrix $\varepsilon_{m,M}$ is regular for $\{(x_i, y_{i,k,j})\}$. It means that the interpolation problem is unique.

3. The Result

3.1. **Univariate Case.** In [2], a Birkhoff interpolation problem was studied. Now, we introduce another case of Birkhoff interpolation problem. In [17], the author introduced Lagrange's fundamental polynomials. For given points $x_0, x_1, ..., x_n$, let us use the fundamental polynomials l_0, $l_1, ..., l_n$, where $l_i(x) = \prod_{j\neq i}(\frac{x - x_j}{x_i - x_j})$ such that

$$l_i(x_k) = \begin{cases} 1 & \text{if} k = i \\ 0 & \text{if} k \neq i \end{cases}, \qquad k, i = 0, 1, ..., n.$$

We recall that the Green's function was defined in [5], [6], [14].

Theorem 6. *Let $\omega_i \in \mathbb{R}^1, i = 0, 1, ..., n$ and $-1 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ and $l_i(x)$ be the fundamental polynomials of Lagrange calculated on the n-1 points $x_i, i = 1, ..., n-1$ and $p_{n,i}(x) = \int_{-1}^{1} G(x,t) l_i(t) dt, i = 1, ..., n-1$, where*

$$G(x,t) = \begin{cases} 1 & t < x \\ 0 & x < t \end{cases}$$

is a Green's function, then the polynomial

$$P_n(x) = \begin{cases} \omega_n & x = x_n \\ \omega_0 + \sum_{i=1}^{n-1} p_{n,i}(x)\omega_i & \text{otherwise} \end{cases} \tag{6}$$

is the unique polynomial of degree $\leq n-1$ which satisfies the Birkhoff interpolation conditions

$$P_n(x_0) = \omega_0,\ P_n'(x_i) = \omega_i,\ i = 1, ..., n-1,\ P_n(x_n) = \omega_n \tag{7}$$

Proof. We know that $P_{n,i}(x)$ is the solution of the boundary value problem

$$\begin{cases} P_{n,i}'(x) = l_i(x) \\ P_{n,i}(-1) = 0 \end{cases}, \quad i = 1, ..., n-1, \quad \text{because} \quad P_{n,i}(x) = \int_{-1}^{x} l_i(t) dt.$$

The polynomial (6) satisfies the interpolatory conditions (7). For the proof of the uniqueness, since $P_{n,i}(x)$ is a polynomial of degree not exceeding n-1, now suppose that $\overline{P}_n$ is another polynomial of degree not exceeding n-1 where it is true in (7) such that $\overline{P}_n(x) \neq P_n(x)$.

We set $\phi_n(x) := \overline{P}_n(x) - P_n(x)$. The polynomial $\phi_n(x)$ has n-1 zeros, therefore it has n-2 optimum, namely, $\phi_n'(x_i) = \overline{P}_n'(x_i) - P_n'(x_i) = 0$. After repeated this process and applying Rolle's theorem, we conclude that $\phi_n(x) \equiv 0$. Thus $\overline{P}_n(x) = P_n(x)$ that is contradiction. □

Remark 1. *By Theorem 6, since the Hermite-Birkhoff interpolation problem with conditions (7) has unique solution then the corresponding Hermite-Birkhoff operator interpolation problem is invariant solvable.*

In [9] and [15], the authors presented Haar approximation. Now, we introduce Haar function and its apply for below example.

Definition 10. The Haar function $\chi_n(x), x \in [0,1]$, where $\chi_1 \equiv 1$, and for $2^k < n \leq 2^{k+1}, k = 0, 1, \cdots$ is defined as follows:

$$\chi_n(x) = \begin{cases} 2^{\frac{k}{2}} & \text{x} \in \Delta_n^+ \\ -2^{\frac{k}{2}} & \text{x} \in \Delta_n^- \\ 0 & \text{x} \notin \overline{\Delta_n} \end{cases} \tag{8}$$

where Δ_n is a binary interval of the form $(\frac{i-1}{2^k}, \frac{i}{2^k})$ where $k = 0, 1, ...$ and $i = 1, 2, ..., 2^k$. For $n = 2^k + i$ we write $\Delta_n = \Delta_k^i = (\frac{i-1}{2^k}, \frac{i}{2^k}), \overline{\Delta_n} = [\frac{i-1}{2^k}, \frac{i}{2^k}], \Delta_1 := \Delta_0^0 = (0,1), \overline{\Delta_1} = [0,1], \Delta_n^+ = (\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}), \Delta_n^- = (\frac{2i-1}{2^{k+1}}, \frac{i}{2^k})$ The values of $\chi_n(x)$ at points of discontinuity and at the endpoints of $[0,1]$ are specified as follows: $\chi_n(x) = \frac{1}{2} lim_{a \to 0}[\chi_n(x+a) + \chi_n(x-a)], x \in (0,1), \chi_n(0) = lim_{t \to 0^+} \chi_n(t)$, $\chi_n(1) = lim_{t \to 0^+} \chi_n(1-t)$.

For clarity of the Theorem 6, we present an example:

Example 1. The solution of Laplace boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < 1 \, , 0 < y < 1 \\ u(0,y) = u(1,y) = 0 \\ u(x,0) = 0 \\ u(x,1) = e^x \end{cases}$$

is $u\,(x,y) = \sum_{n=1}^{\infty} b_n \sinh{(n\pi y)} \sin{(n\pi x)}$ where $b_n = \frac{2n\pi - 2ne\pi(-1)^n}{(1+n^2\pi^2) sinh(n\pi)}$.

Using theorem 6, we compute Birkhoff interpolation polynomial for $f(x) = e^x$ in these knots: $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$. Let $\omega_0 = f(0) = 1, \omega_1 = f'(\frac{1}{4}) = e^{\frac{1}{4}}, \omega_2 = f'(\frac{1}{2}) = e^{\frac{1}{2}}, \omega_3 = f'(\frac{3}{4}) = e^{\frac{3}{4}}, \omega_4 = f(1) = e$ then, $l_1(t) = 8t^2 - 10t + 3, l_2(t) = -16t^2 + 16t - 3, l_3(t) = 8t^2 - 6t + 1$ are Lagrange polynomials on x_1, x_2, x_3 and $P_{4,1}(x) = \frac{8}{3}x^3 - 5x^2 + 3x, P_{4,2}(x) = \frac{-16}{3}x^3 + 8x^2 + 3x, P_{4,3}(x) = \frac{8}{3}x^3 - 3x^2 + x$ thus $P_4(x) = (\frac{8}{3}e^{\frac{1}{4}} - \frac{16}{3}e^{\frac{1}{2}} + \frac{8}{3}e^{\frac{3}{4}})x^3 + (-5e^{\frac{1}{4}} + 8e^{\frac{1}{2}} - 3e^{\frac{3}{4}})x^2 + (3e^{\frac{1}{4}} - 3e^{\frac{1}{2}} + e^{\frac{3}{4}})x + 1$ and also $p_4(1) = e$.

Now, we employ approximation Haar-Fourier $P_H(x) = \sum_{n=1}^{\infty} C_n(f)\chi_n(x)$ for the function $f(x) = e^x$ and its compare to $P_4(x)$. First, we compute Haar-Fourier coefficients $C_n(f) = \int_0^1 f(x)\chi_n(x)dx$ as follows: $C_1(f) = e - 1, C_2(f) = 2e^{1/2} - e - 1, C_3(f) = 2\sqrt{2}e^{1/4} - \sqrt{2}e^{1/2} - \sqrt{2}, C_4(f) = 2\sqrt{2}e^{3/4} - \sqrt{2}e^{1/2} - \sqrt{2}e$, thus the Haar polynomial is: $P_H(x) = e - 1 + (2e^{1/2} - e - 1)\chi_2(x) + (2\sqrt{2}e^{1/4} - \sqrt{2}e^{1/2} - \sqrt{2})\chi_3(x) + (2\sqrt{2}e^{3/4} - \sqrt{2}e^{1/2} - \sqrt{2}e)\chi_4(x)$ Using the following ten points, we compare $f(x), P_4(x), P_H(x)$

TABL. 1. Comparison of $f(x)$, $p_4(x)$, $p_H(x)$ in $[0,1]$

x	$f(x)$	$p_4(x)$	$p_H(x)$
0	1	1	1.136101666
0.1	1.105170918	1.106753896	1.136101666
0.25	1.284025417	1.286209257	1.297442541
0.3	1.349858808	1.352009578	1.458783416
0.5	1.648721271	1.650644616	1.665949200
0.7	2.013752707	2.020817622	1.873114984
0.75	2.117000017	2.119201800	2.139121116
0.9	2.459603111	2.461087206	2.405127248
0.99	2.691234472	2.691012369	2.405127248
1	2.718281828	2.717776531	2.405127248

Consequently one might favor Birkhoff interpolation in some cases.
Now, we set $p_4(x)$ instead of $f(x) = e^x$ in Laplace boundary value problem and obtain the approximation solution $u(x,y) = \Sigma_{n=1}^{\infty} a_n \sinh(n\pi y)\sin(n\pi x)$ where $a_n = \frac{2}{\sinh(n\pi)} \int_0^1 p_4(x)\sin(n\pi x)dx$.

Using Maple program, graphs are as follows:

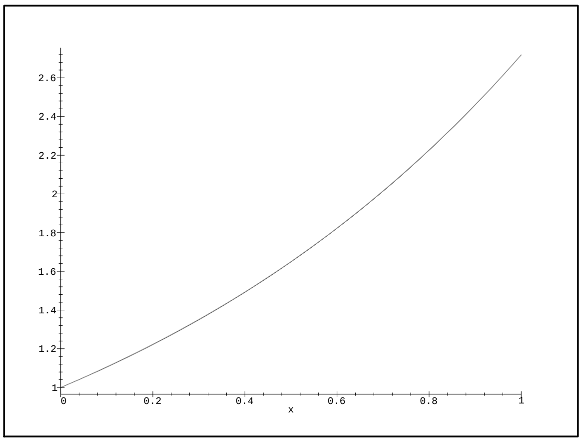

FIG. 1. Comparison of $f(x)$ with $p_4(x)$ in [0,1]

3.2. **Bivariate Case.** In this paper, uniqueness is investigated in another way. In [7, Corollary 3.4, p.27], if the incidence matrix is characterized, then the interpolation polynomial can be obtained.

Now, suppose that the interpolation conditions are given. Then, we compute a corresponding matrix as follows:

Theorem 7. *Suppose that the bivariate Birkhoff interpolation problem (5) is given. Then, the incidence matrix is characterized.*

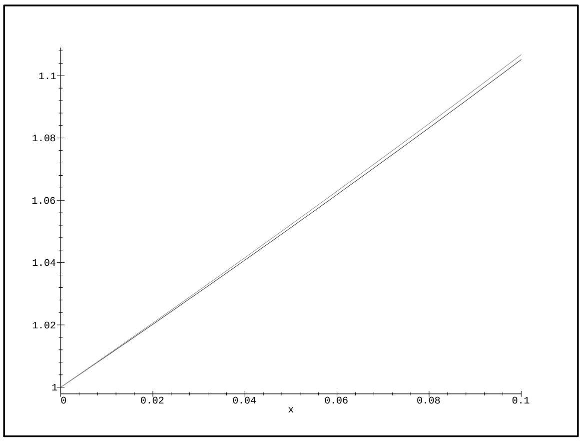

FIG. 2. Comparison of $f(x)$ with $p_4(x)$ in [0,0.01]

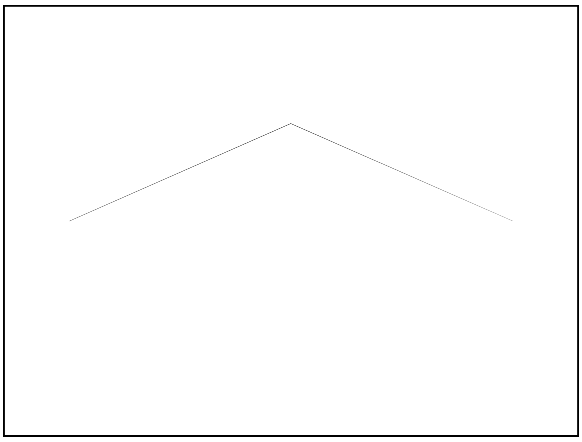

FIG. 3. Graph $u\,(x,y)=\sum_{n=1}^{1000} b_n \sinh\,(n\pi y)\sin\,(n\pi x)$

Proof. First, we arrange the knots as follows $x_1 < ... < x_m$, where for each the second component of these points, namely, $y_{i,k,1},...,y_{i.k.j}$, $1 \le j \le a_{i,k}$, where $a_{i,k} \in \mathbb{N}$.

Note that k is the order of partial derivative of the first variable for P(x,y), and we denote the order of partial derivative of second variable for P(x,y) by l, where $0 \le l \le N_{i,k}$, $N_{i,k} \in \mathbb{N}_0$.

Let Z be a set of pairwise (i,k)'s in (5). For indices i,j,k,l in (5), we define $e_{i,j}^{k,l} = 1, (i,k) \in Z$.

Using $e_{i,j}^{k,l}$, we construct a matrix where j,l are the number of rows and columns, respectively.

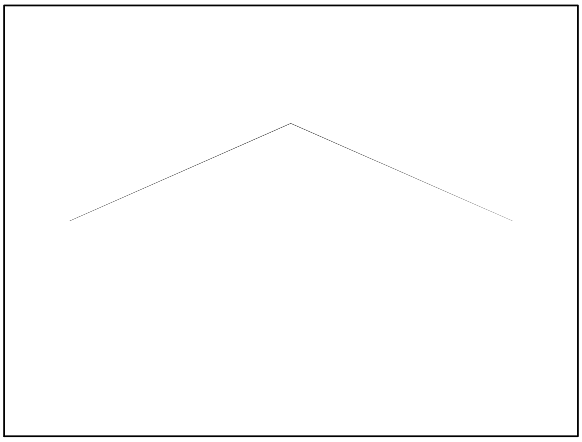

FIG. 4. Graph $u\,(x, y) = \sum_{n=1}^{1000} a_n \sinh\,(n\pi y) \sin\,(n\pi x)$

Let $E_{i,k} = (e_{i,j}^{k,l})_{j=1}^{a_{i,k}}{}_{,l=0}^{N_{i,k}}, (i,k) \in Z$. Regarding $E_{i,k}$, the number of rows and columns are equal $a_{i,k}$ and $N_{i,k} + 1$ respectively. It means that for every $(i,k) \in Z$ the value of $e_{i,j}^{k,l}$ is equal 1 otherwise is equal 0. But for the other points $(i,k) \in \{1, ..., m\} \times \{0, ..., M\}$ every array of $E_{i,k}$ equals zero where

$$M = |Z| - 1 \tag{9}$$

Thus, for the bivariate Birkhoff interpolation problem (5), the corresponding matrix is $\varepsilon_{m,M} = (E_{i,k})_{i=1}^{m}{}_{,k=0}^{M}$ that is an incidence matrix. □

Now, we present two examples as follows and apply Theorem 7 to obtain interpolation polynomial. In the first example, we use incidence matrix and obtain interpolation polynomial. In the second example, we use interpolation conditions and obtain interpolation polynomial.

Example 2. Consider bivariate incidence matrix

$$\varepsilon_{4,4} = \begin{Vmatrix} 0 & \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} & 0 & 0 & 0 \\ \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix} & 0 & 0 & 0 & 0 \\ 0 & \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} & \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix} & 0 & 0 \\ \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix} & 0 & 0 & 0 & 0 \end{Vmatrix} \tag{10}$$

In view of Theorem 7, we have

$$Z = \{(1,1), (2,0), (3,1), (3,2), (4,0)\}, \quad m = M = 4,$$

$$E_{1,1} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \Rightarrow a_{1,1} = 3, \quad N_{1,1} = 3,$$

$$E_{2,0} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix} \Rightarrow a_{2,0} = 3, \quad N_{2,0} = 4,$$

$$E_{3,1} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \Rightarrow a_{3,1} = 4, \quad N_{3,1} = 3,$$

$$E_{3,2} = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix} \Rightarrow a_{3,2} = 2, \quad N_{3,2} = 1,$$

$$E_{4,0} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \Rightarrow a_{4,0} = 2, \quad N_{4,0} = 14$$

Using (3.6) in [7], we have

$N_1 = N_{2,0} = 4$, $M_1 = 0$

$N_2 = N_{1,1} = N_{3,1} = 3$, $M_2 = 2$

$N_3 = N_{3,2} = N_{4,0} = 1$, $M_3 = 4$

Consider the following points in $[0,1]^2$

$$\begin{cases} x_1 = 0, x_2 = 0.1, x_3 = 0.9, x_4 = 1 \\ y_{1,1,1} = 0, y_{1,1,2} = 0.1, y_{1,1,3} = 0.2 \\ y_{2,0,1} = 0, y_{2,0,2} = 0.2, y_{2,0,3} = 0.5 \\ y_{3,1,1} = 0, y_{3,1,2} = 0.3, y_{3,1,3} = 0.6, y_{3,1,4} = 0.9 \\ y_{3,2,1} = 0.8, y_{3,2,2} = 1 \\ y_{4,0,1} = 0, y_{4,0,2} = 1 \end{cases} \tag{11}$$

Since the incidence matrices $E_{i,k}$'s are regular and

$$E_1 = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad E_3 = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

are also regular, then by [7, Corollary 3.4,p.27], the incidence matrix $\varepsilon_{4,4}$ is regular. So, bivariate Birkhoff interpolation problem

$(C^q([0,1]^2), \Pi_0 \otimes \Pi_4 + \Pi_2 \otimes \Pi_3 + \Pi_4 \otimes \Pi_1; D^{k,l}_{x_i, y_{i,k,j}} : (i,k) \in Z, (x_i, y_{i,k,j}) \in T)$,

where

$q = max\{M_s + N_s\}_{s=1}^{3} = 5$, and $x_1 < ... < x_4, y_{i,k,1} < ... < y_{i,k,a_{i,k}}$

is uniquely solvable.

That is for all $f \in C^5([0,1]^2)$ for example $f(x,y) = ye^x$ there exists

$$P \in \sum_{s=1}^{3} \Pi_{M_s} \otimes \Pi_{N_s}$$

i.e.

$P(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy +$

$+a_{0,2}y^2 + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + a_{4,0}x^4 +$
$+a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + a_{4,1}x^4y + a_{2,3}x^2y^3$
and

$$\begin{cases} \frac{\partial^3 P}{\partial y^2 \partial x}(0,0) = \frac{\partial^3 f}{\partial y^2 \partial x}(0,0) \\ \frac{\partial P}{\partial x}(0,0.1) = \frac{\partial f}{\partial x}(0,0.1) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(0,0.1) = \frac{\partial^3 f}{\partial y^2 \partial x}(0,0.1) \\ \frac{\partial^2 P}{\partial y \partial x}(0,0.2) = \frac{\partial^2 f}{\partial y \partial x}(0,0.2) \end{cases}, \quad \begin{cases} P(0.1,0) = f(0.1,0) \\ \frac{\partial^2 P}{\partial y^2}(0.1,0) = \frac{\partial^2 f}{\partial y^2}(0.1,0) \\ \frac{\partial P}{\partial y}(0.1,0.2) = \frac{\partial f}{\partial y}(0.1,0.2) \\ \frac{\partial^2 P}{\partial y^2}(0.1,0.2) = \frac{\partial^2 f}{\partial y^2}(0.1,0.2) \\ \frac{\partial P}{\partial y}(0.1,0.5) = \frac{\partial f}{\partial y}(0.1,0.5) \end{cases}$$

$$\begin{cases} \frac{\partial P}{\partial x}(0.9,0) = \frac{\partial f}{\partial x}(0.9,0) \\ \frac{\partial^2 P}{\partial y \partial x}(0.9,0.3) = \frac{\partial^2 f}{\partial y \partial x}(0.9,0.3) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(0.9,0.6) = \frac{\partial^3 f}{\partial y^2 \partial x}(0.9,0.6) \\ \frac{\partial^4 P}{\partial y^3 \partial x}(0.9,0.9) = \frac{\partial^4 f}{\partial y^3 \partial x}(0.9,0.9) \end{cases}, \quad \begin{cases} \frac{\partial^2 P}{\partial x^2}(0.9,0.8) = \frac{\partial^2 f}{\partial x^2}(0.9,0.8) \\ \frac{\partial^2 P}{\partial x^2}(0.9,0.1) = \frac{\partial^2 f}{\partial x^2}(0.9,0.1) \end{cases},$$

$$\begin{cases} P(1,0) = f(1,0) \\ \frac{\partial P}{\partial y}(1,1) = \frac{\partial f}{\partial y}(1,1) \end{cases}$$

By the conditions above, the algebraic system of coefficients of p(x,y) is as follows:

$$\begin{cases} 2a_{1,2} = 0 \\ a_{1,0} + 0.1a_{1,1} + 0.01a_{1,2} + 0.001a_{1,3} = 0.1 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ 1 + a_{0,1} + a_{2,1} + 3a_{0,3} + a_{3,1} + 4a_{0,4} + a_{4,1} = e \end{cases}$$

Therefore, the solution of system is
$a_{1,0} = a_{0,2} = a_{1,2} = a_{0,3} = a_{2,2} = a_{1,3} = a_{0,4} = a_{2,3} = 0, a_{1,1} = 1,$
$a_{0,0} = 3.46 \times 10^{-7}, \quad a_{0,1} = 0.999928024, \quad a_{4,1} = 6.678685615,$
$a_{3,1} = 0.13497021, \quad a_{2,1} = 0.510326531, \quad a_{2,0} = -1.314 \times 10^{-5},$
$a_{4,0} = -1.5909 \times 10^{-5}, \quad a_{3,0} = 2.8702 \times 10^{-5}$
Thus, the Birkhoff polynomial P is
$P_B(x,y) = 0.0000003459603111 + 0.999928024y - 0.00001314x^2 +$
$+xy + 0.000028702x^3 + 0.510326531x^2y -$
$-0.000015909x^4 + 0.13497021x^3y + 6.678685615x^4y.$
In the following example, the knots and Birkhoff conditions are given then, we obtain Birkhoff polynomial.

Example 3. By the following knots in $[0,1]^2$ and Birkhoff conditions and in view of Theorem 7 and the indices i,j,k,l, we have

$$\begin{cases} \frac{\partial^3 P}{\partial y^2 \partial x}(x_1,y_1) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_1,y_1) \\ \frac{\partial P}{\partial x}(x_1,y_2) = \frac{\partial f}{\partial x}(x_1,y_2) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(x_1,y_2) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_1,y_2) \\ \frac{\partial^2 P}{\partial y \partial x}(x_1,y_3) = \frac{\partial^2 f}{\partial y \partial x}(x_1,y_3) \end{cases}, \quad \begin{cases} P(x_2,y_1) = f(x_2,y_1) \\ \frac{\partial^2 P}{\partial y^2}(x_2,y_1) = \frac{\partial^2 f}{\partial y^2}(x_2,y_1) \\ \frac{\partial P}{\partial y}(x_2,y_2) = \frac{\partial f}{\partial y}(x_2,y_2) \\ \frac{\partial^2 P}{\partial y^2}(x_2,y_2) = \frac{\partial^2 f}{\partial y^2}(x_2,y_2) \\ \frac{\partial P}{\partial y}(x_2,y_3) = \frac{\partial f}{\partial y}(x_2,y_3) \end{cases}$$

$$\begin{cases} \frac{\partial P}{\partial x}(x_3, y_1) = \frac{\partial f}{\partial x}(x_3, y_1) \\ \frac{\partial^2 P}{\partial y \partial x}(x_3, y_2) = \frac{\partial^2 f}{\partial y \partial x}(x_3, y_2) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(x_3, y_3) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_3, y_3) \\ \frac{\partial^4 P}{\partial y^3 \partial x}(x_3, y_4) = \frac{\partial^4 f}{\partial y^3 \partial x}(x_3, y_4) \end{cases}, \quad \begin{cases} \frac{\partial^2 P}{\partial x^2}(x_3, y_1) = \frac{\partial^2 f}{\partial x^2}(x_3, y_1) \\ \frac{\partial^2 P}{\partial x^2}(x_3, y_2) = \frac{\partial^2 f}{\partial x^2}(x_3, y_2) \end{cases},$$

$$\begin{cases} P(x_4, y_1) = f(x_4, y_1) \\ \frac{\partial P}{\partial y}(x_4, y_2) = \frac{\partial f}{\partial y}(x_4, y_2) \end{cases}$$

Now, we consider the points (11) in $[0,1]^2$, then

$$Z = \{(1,1), (2,0), (3,1), (3,2), (4,0)\}.$$

Regularity of $E_{i,k}$'s is obvious here:

$$E_{1,1} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \quad E_{2,0} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix}, \quad E_{3,1} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$E_{3,2} = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix}, \quad E_{4,0} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

Therefore,

$$a_{1,1} = 3, \quad N_{1,1} = 3, \quad a_{2,0} = 3, \quad N_{2,0} = 4$$

$$a_{3,1} = 4, \quad N_{3,1} = 3, \quad a_{3,2} = 2, \quad N_{3,2} = 1$$

$$a_{4,0} = 2, \quad N_{4,0} = 1$$

and also

$$N_1 = N_{2,0} = 4, \quad M_1 = 0,$$

$$N_2 = N_{1,1} = N_{3,1} = 3, \quad M_2 = 2$$

$$N_3 = N_{3,2} = N_{4,0} = 1, \quad M_3 = 4.$$

Using (9), we can write incidence matrix $\varepsilon_{4,4}$ in (10).
Thus, matrices

$$E_1 = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad E_3 = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

for knots x_1, x_2, x_3, x_4 and also the incidence matrices $E_{1,1}, E_{2,0}, E_{3,1}, E_{3,2}, E_{4,0}$ for knots $y_{i,k,j}$'s are regular. Thus by corollary 3.4 of [7, P.27], $\varepsilon_{4,4}$ is regular. For every $f \in C^5([0,1]^2)$ there exists $P \in \sum_{s=1}^{3} \Pi_{M_s} \otimes \Pi_{N_s}$ so that it satisfies interpolation conditions. Finally, with knots $(x_i, y_{i,k,j})$ in (11), we can establish P.

3.3. **Bivariate Haar Approximation.** In [9] and [15], the authors presented univariate Haar series. Now, we investigate a new case of bivariate Haar approximation in the following example.

Example 4. Using the approximation presented in [4], we compute Haar – Fourier coefficients

$$a_{m,n}(f) = \int_0^1 \int_0^1 f(x,y)\chi_{m,n}(x,y)dxdy$$

for bivariate function $f(x,y) = ye^x$ then, by (20) in [4], we have

$$a_{1,1}(f) = \frac{e-1}{2}, \quad a_{2,2}(f) = \frac{2e^{1/2}-e-1}{2},$$

$$a_{3,3}(f) = \frac{2e^{1/4}-e^{1/2}-1}{4}, \quad a_{3,4}(f) = \frac{6e^{1/4}-3e^{1/2}-3}{4},$$

$$a_{4,3}(f) = \frac{2e^{3/4}-e^{1/2}-e}{4}, \quad a_{4,4}(f) = \frac{6e^{3/4}-3e^{1/2}-3e}{4}.$$

We recall that the Haar function is given

$$\chi_{m,n}(x,y) = \begin{cases} 2^k & x \in \Delta_m^+, y \in \Delta_n \\ -2^k & x \in \Delta_m^-, y \in \Delta_n \\ 0 & (x,y) \notin \overline{\Delta_{n,m}} \end{cases}$$

where $\chi_{1,1} \equiv 1$ and the binary interval Δ_n and other signs in Definition 10 are satisfied. Then the Haar polynomial is:

$$P_H(x,y) = 2.218281828 - 0.210419644\chi_{2,2}(x,y) + 1.919329563\chi_{3,3}(x,y) +$$

$$+0.486540953\chi_{3,4}(x,y) - 0.033250766\chi_{4,3}(x,y) - 0.099752299\chi_{4,4}(x,y).$$

4. Comparison of function $f(x,y) = ye^x$ with $P_B(x,y)$ and $P_H(x,y)$

Using the following eight points, we compare $f(x,y), P_B(x,y), P_H(x,y)$

Tabl. 2. comparison of $f(x,y), P_B(x,y), P_H(x,y)$

(x,y)	$f(x,y)$	$p_B(x,y)$	$p_H(x,y)$
(0,0)	0	0.0000003	5.846521
(0,0.1)	0.1	0.099993	5.846521
(0.1,0)	0	0.0000002	5.846521
(0.1,0.1)	0.110517	0.110583	5.846521
(0.2,0.5)	0.610701	0.616053	4.413732
(0.9,0.1)	0.245960	0.679357	2.495203
(0.2,0.9)	1.099262	1.108896	2.980944
(0.9,0.9)	2.213642	6.114214	2.628206

Now, we compare $f(x,y), P_B(x,y)$ by using their graphs.

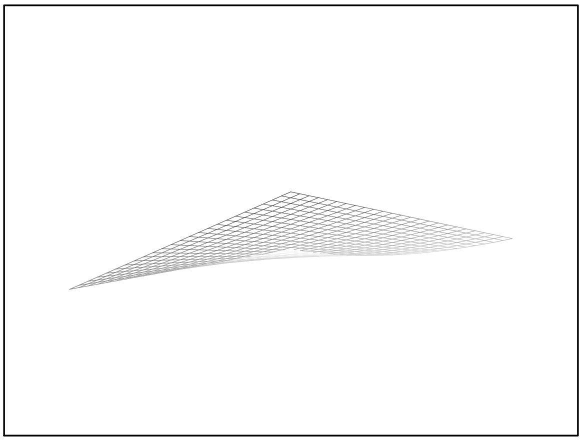

FIG. 5. The graph of $f(x,y)$ on $[0,1]^2$

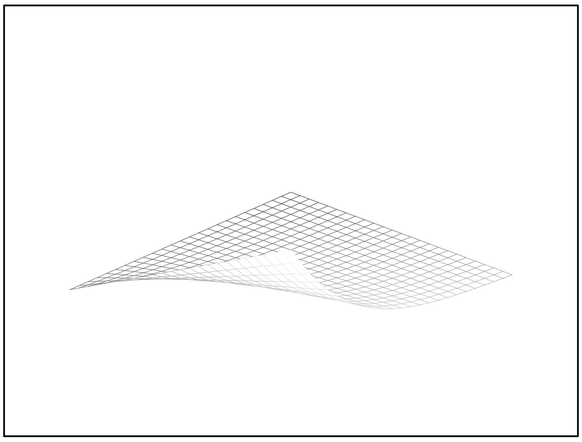

FIG. 6. The graph of $P_B(x,y)$ on $[0,1]^2$

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RESONANT LIQUID SLOSHING IN AN UPRIGHT CIRCULAR TANK PERFORMING A PERIODIC MOTION

I. A. Raynovskyy, A. N. Timokha

Резюме. Використовується слабо-нелінійна модальна теорія Нарімано-ва-Моісєєва для аналізу усталених резонансних хвиль в вертикальному циліндричному резервуарі, який рухається періодично з частотою, близькою до першої власної частоти коливання рідини.

Abstract. A weakly-nonlinear Narimanov-Moiseev type modal theory is used to analyse steady-state resonant waves in an upright circular tank which moves periodically with the forcing frequency close to the lowest natural sloshing frequency.

1. Introduction

The upright circular tank is relevant for spacecraft applications, the pressure-suppression pools of Boiling Water Reactors, storage tanks, Tuned Liquid Dampers, offshore towers, and basins of the aqua-cultural engineering. Resonant sloshing due to harmonic excitations of the tank was extensively studied, theoretically and experimentally, in [1,3,4,6]. For the longitudinal tank forcing, steady-state planar (in the excitation plane), swirling and irregular (chaotic) waves were detected [1, 4, 6] when the forcing frequency is close to the lowest natural sloshing frequency. A review on sloshing due to parametric (vertical) excitations is given in [3]. However, the above-mentioned industrial applications deal, normally, with the coupled rigid tank-and-sloshing dynamics when the tank performs complex three-dimensional motions which unnecessarily occur in either meridional plane or vertical direction. This causes an interest to analytical studies on the resonant steady-state sloshing due to a three-dimensional periodic tank excitation that are done in the present paper by employing the weakly-nonlinear modal system [7].

2. Statement of the problem

An inviscid incompressible contained liquid with irrotational flows sloshes in an upright circular rigid tank with radius r_0. The tank performs small-magnitude prescribed periodic sway, surge, roll, and pitch motions which are described by the r_0-scaled generalised coordinates $\eta_1(t)$ and $\eta_2(t)$ (horizontal tank motions) and angular perturbations $\eta_4(t)$ and $\eta_5(t)$ (see, figure 1). The yaw cannot excite sloshing within the framework of the inviscid potential flow model but the heave is not considered. All geometric and physical parameters

Key words. Sloshing, multimodal method; periodic solution; response curves.

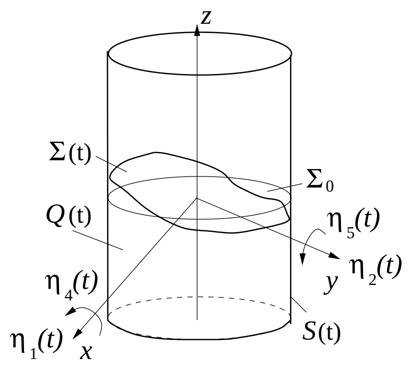

FIG. 1. The time-dependent liquid domain $Q(t)$ confined by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$. The free-surface evolution is considered in the tank-fixed coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the mean (hydrostatic) free surface Σ_0 and Oz is the symmetry axis. Small-magnitude periodic tank motions are governed by the generalised coordinates $\eta_1(t)$ (surge), $\eta_4(t)$ (roll), $\eta_2(t)$ (sway), and $\eta_5(t)$ (pitch). The mean free surface Σ_0 is perpendicular to Oz

are henceforth considered *scaled* by r_0. We introduce a small parameter $0 < \epsilon \ll 1$ characterising the periodic forcing, i.e. $\eta_i(t) = O(\epsilon)$, $i = 1, 2, 4, 5$.

Figure 1 shows the time-dependent liquid domain $Q(t)$ with the free surface $\Sigma(t)$ (governed by the single-valued function $z = \zeta(r, \theta, t)$) and the wetted tank surface $S(t)$. The liquid flow is determined by the velocity potential $\Phi(r, \theta, z, t)$. The unknowns, ζ and Φ, are defined in the tank-fixed Cartesian (equivalent cylindrical) non-inertial coordinate system; they can be found from either the corresponding free-surface problem or its equivalent variational formulation. The latter formulation facilitates the multimodal method, which employs the Fourier-type representations of ζ and Φ in which the time-dependent coefficients are interpreted as *generalised coordinates and velocities.* The representations are normally based on the natural sloshing modes which are the eigenfunctions of the spectral boundary problem

$$\nabla^2\varphi = 0 \text{ in } Q_0, \quad \frac{\partial \varphi}{\partial n} = 0 \text{ on } S_0, \quad \frac{\partial \varphi}{\partial n} = \kappa\varphi \text{ on } \Sigma_0, \quad \int_{\Sigma_0} \varphi \, \mathrm{d}S = 0 \qquad (1)$$

in the mean (hydrostatic) liquid domain Q_0 confined by the mean free surface Σ_0 and the wetted tank surface S_0. The r_0-scaled problem (1) has the analytical solution [4]

$$\varphi_{Mi}(r, z, \theta) = \mathcal{R}_{Mi}(r)\, \mathcal{Z}_{Mi}(z) \frac{\cos M\theta}{\sin M\theta}, \quad M = 0, \ldots; \; i = 1, \ldots, \qquad (2a)$$

$$\mathcal{R}_{Mi}(r) = \alpha_{Mi} J_M(k_{Mi} r), \quad \mathcal{Z}_{Mi}(z) = \frac{\cosh(k_{Mi}(z + h))}{\cosh(k_{Mi} h)}, \qquad (2b)$$

where $J_M(\cdot)$ is the Bessel functions of the first kind, the radial wave numbers k_{Mi} are determined by $\mathcal{R}'_{M,i}(r_1) = 0$ and the normalising multipliers α_{Mi} follow from the orthogonality condition

$$\lambda_{(Mi)(Mj)} = \int_{r_1}^{1} r\,\mathcal{R}_{Mi}(r)\,\mathcal{R}_{Mj}(r)\,\mathrm{d}r = \delta_{ij}, \quad i,j = 1,\dots, \tag{3}$$

where δ_{ij} is the Kronecker delta. The eigenvalues κ_{Mi} and the natural sloshing frequencies σ_{Mi} read as

$$\kappa_{Mi} = k_{Mi}\tanh(k_{Mi}h) \text{ and } \sigma^2_{Mi} = \kappa_{Mi}\,\bar{g}/r_0 = \kappa_{Mi}\,g, \tag{4}$$

respectively, where $\bar{g}$ is the dimensional gravity acceleration.

Dealing with a small-amplitude angular tank motion requires the linearised Stokes-Joukowski potentials $\Omega_{0i}(r,z,\theta)$, $i = 1,2,3$ which are harmonic functions satisfying the Neumann boundary conditions

$$\frac{\partial\Omega_{01}}{\partial n} = -(zn_r - rn_z)\sin\theta, \ \frac{\partial\Omega_{02}}{\partial n} = (zn_r - rn_z)\cos\theta, \ \frac{\partial\Omega_{03}}{\partial n} = 0 \tag{5}$$

on Σ_0 and the wetted tank surface S_0, where n_r and n_z are the outer normal components in the r- and z- directions. This implies $\Omega_{01} = -F(r,z)\sin\theta$, $\Omega_{02} = F(r,z)\cos\theta$, $\Omega_{03} = 0$, where

$$\begin{gathered} F(r,z) = rz + \sum_{n=1}^{\infty} -\frac{2\,P_n}{k_{1n}}\,\mathcal{R}_{1n}(r)\frac{\sinh(k_{1n}(z+\frac{1}{2}h))}{\cosh(\frac{1}{2}k_{1n}h)}, \\ P_n = \int_{r_1}^{1} r^2\,\mathcal{R}_{1n}(r)\,\mathrm{d}r. \end{gathered} \tag{6}$$

When adopting (2a) and (6), the *aforementioned Fourier (modal) representation* takes the form [7]

$$\zeta(r,\theta,t) = \sum_{M,i}^{I_\theta,I_r} \mathcal{R}_{Mi}(r)\cos(M\theta)\,p_{Mi}(t) + \sum_{m,i}^{I_\theta,I_r} \mathcal{R}_{mi}(r)\sin(m\theta)\,r_{mi}(t), \tag{7a}$$

$$\begin{aligned} \Phi(r,\theta,z,t) &= \dot{\eta}_1(t)\,r\cos\theta + \dot{\eta}_2(t)\,r\sin\theta + \\ &+ F(r,z)[-\dot{\eta}_4(t)\sin\theta + \dot{\eta}_5(t)\cos\theta] + \\ &+ \sum_{M,i}^{I_\theta,I_r} \mathcal{R}_{Mi}(r)\,\mathcal{Z}_{Mi}(z)\,\cos(M\theta)\,P_{Mi}(t) + \\ &+ \sum_{m,i}^{I_\theta,I_r} \mathcal{R}_{mi}(r)\,\mathcal{Z}_{mi}(z)\,\sin(m\theta)\,R_{mi}(t), \end{aligned} \tag{7b}$$

I_θ, $I_r \to \infty$. *Here and further*, all capital summation letters imply changing from zero to I_θ but the lower case indices mean changing from one to either I_θ or I_r.

In the modal representation (7), p_{Mi} and r_{mi} play the role of the sloshing-related generalised coordinates but P_{Mi} and R_{mi} are the corresponding generalised velocities. Using the Bateman-Luke variational formulation makes it possible to derive the Euler-Lagrange equations with respect to the generalised coordinates and velocities. The procedure is described in [7] where the latter equations are explicitly written down in both fully- and weakly-nonlinear forms.

The weakly-nonlinear equations are constructed in [7] adopting the Narimanov-Moiseev asymptotic relations

$$p_{11} \sim r_{11} = O(\epsilon^{1/3}), \quad p_{0j} \sim p_{2j} \sim r_{2j} = O(\epsilon^{2/3}),$$
$$r_{1(j+1)} \sim p_{1(j+1)} \sim p_{3j} \sim r_{3j} = O(\epsilon), \quad j = 1, 2, \ldots, I_r; \quad I_r \to \infty \quad (8)$$

(see, an extensive discussion on what these relations mean for axisymmetric tanks in [5]). The equations take the form

$$\ddot{p}_{11} + \sigma_{11}^2 p_{11} + d_1 p_{11}\left(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right) + d_2\left[r_{11}(\ddot{p}_{11}r_{11} - \ddot{r}_{11}p_{11}) + 2\dot{r}_{11}(\dot{p}_{11}r_{11} - \dot{r}_{11}p_{11})\right]$$
$$+ \sum_{j=1}^{I_r}\Big[d_3^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j} + \dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_4^{(j)}(\ddot{p}_{2j}p_{11} + \ddot{r}_{2j}r_{11})$$
$$+ d_5^{(j)}\left(\ddot{p}_{11}p_{0j} + \dot{p}_{11}\dot{p}_{0j}\right) + d_6^{(j)}\ddot{p}_{0j}p_{11}\Big] = -(\ddot{\eta}_1 - g\eta_5 - S_1\ddot{\eta}_5)\kappa_{11}P_1, \quad (9a)$$

$$\ddot{r}_{11} + \sigma_{11}^2 r_{11} + d_1 r_{11}\left(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right) + d_2\left[p_{11}(\ddot{r}_{11}p_{11} - \ddot{p}_{11}r_{11}) + 2\dot{p}_{11}(\dot{r}_{11}p_{11} - \dot{p}_{11}r_{11})\right]$$
$$+ \sum_{j=1}^{I_r}\Big[d_3^{(j)}\left(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j} + \dot{p}_{11}\dot{r}_{2j} - \dot{p}_{2j}\dot{r}_{11}\right) + d_4^{(j)}(\ddot{r}_{2j}p_{11} - \ddot{p}_{2j}r_{11})$$
$$+ d_5^{(j)}(\ddot{r}_{11}p_{0j} + \dot{r}_{11}\dot{p}_{0j}) + d_6^{(j)}\ddot{p}_{0j}r_{11}\Big] = -(\ddot{\eta}_2 + g\eta_4 + S_1\ddot{\eta}_4)\kappa_{11}P_1; \quad (9b)$$

$$\ddot{p}_{2k} + \sigma_{2k}^2 p_{2k} + d_{7,k}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + d_{9,k}(\ddot{p}_{11}p_{11} - \ddot{r}_{11}r_{11}) = 0, \quad (10a)$$
$$\ddot{r}_{2k} + \sigma_{2k}^2 r_{2k} + 2d_{7,k}\dot{p}_{11}\dot{r}_{11} + d_{9,k}(\ddot{p}_{11}r_{11} + \ddot{r}_{11}p_{11}) = 0, \quad (10b)$$
$$\ddot{p}_{0k} + \sigma_{0k}^2 p_{0k} + d_{8,k}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{10,k}(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11}) = 0; \quad (10c)$$

$$\ddot{p}_{3k} + \sigma_{3k}^2 p_{3k} + d_{11,k}\left[\ddot{p}_{11}(p_{11}^2 - r_{11}^2) - 2p_{11}r_{11}\ddot{r}_{11}\right] + d_{12,k}\left[p_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) - 2r_{11}\dot{p}_{11}\dot{r}_{11}\right]$$
$$+ \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}p_{2j} - \ddot{r}_{11}r_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j}p_{11} - \ddot{r}_{2j}r_{11})$$
$$+ d_{15,k}^{(j)}(\dot{p}_{2j}\dot{p}_{11} - \dot{r}_{2j}\dot{r}_{11})\Big] = 0, \quad (11a)$$

$$\ddot{r}_{3k} + \sigma_{3k}^2 r_{3k} + d_{11,k}\left[\ddot{r}_{11}(p_{11}^2 - r_{11}^2) + 2p_{11}r_{11}\ddot{p}_{11}\right] + d_{12,k}\left[r_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + 2p_{11}\dot{p}_{11}\dot{r}_{11}\right]$$
$$+ \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}r_{2j} + \ddot{r}_{11}p_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j}r_{11} + \ddot{r}_{2j}p_{11})$$
$$+ d_{15,k}^{(j)}(\dot{p}_{2j}\dot{r}_{11} + \dot{r}_{2j}\dot{p}_{11})\Big] = 0, \quad k = 1, ..., I_r; \quad (11b)$$

$$\ddot{p}_{1n} + \sigma_{1n}^2 p_{1n} + d_{16,n}(\ddot{p}_{11}p_{11}^2 + r_{11}p_{11}\ddot{r}_{11}) + d_{17,n}(\ddot{p}_{11}r_{11}^2 - r_{11}p_{11}\ddot{r}_{11})$$
$$+ d_{18,n}p_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(r_{11}\dot{p}_{11}\dot{r}_{11} - p_{11}\dot{r}_{11}^2)$$
$$+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j})$$
$$+ d_{22,n}^{(j)}(\dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_{23,n}^{(j)}\ddot{p}_{11}p_{0j} + d_{24,n}^{(j)}p_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{p}_{11}\dot{p}_{0j}\Big]$$
$$= -(\ddot{\eta}_1 - g\eta_5 - S_n\ddot{\eta}_5)\kappa_{1n}P_n, \quad (12a)$$

$$\ddot{r}_{1n} + \sigma_{1n}^2 r_{1n} + d_{16,n}(\ddot{r}_{11}r_{11}^2 + r_{11}p_{11}\ddot{p}_{11}) + d_{17,n}(\ddot{r}_{11}p_{11}^2 - r_{11}p_{11}\ddot{p}_{11})$$
$$+ d_{18,n}r_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(p_{11}\dot{p}_{11}\dot{r}_{11} - r_{11}\dot{p}_{11}^2)$$
$$+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{r}_{2j} - r_{11}\ddot{p}_{2j})$$
$$+ d_{22,n}^{(j)}(\dot{p}_{11}\dot{r}_{2j} - \dot{r}_{11}\dot{p}_{2j}) + d_{23,n}^{(j)}\ddot{r}_{11}p_{0j} + d_{24,n}^{(j)}r_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{r}_{11}\dot{p}_{0j}\Big]$$
$$= -(\ddot{\eta}_2 + g\eta_4 + S_n\ddot{\eta}_4)\kappa_{1n}P_n, \quad n = 2, ..., I_r. \quad (12b)$$

They couple all generalised coordinates up to the $O(\epsilon)$-order as $I_r \to \infty$; $r_{kl} \sim p_{kl} = o(\epsilon)$, $k \geq 4$ and, therefore, are neglected. The hydrodynamic coefficients of (9)–(12) are functions of the nondimensional liquid depth h. The system needs either initial or periodicity condition that determines transient and steady-state solutions, respectively.

3. Steady-state (periodic) resonant solutions

Applicability of (9)–(12) for studying the steady-state (periodic) waves requires that

- the generalised coordinates $\eta_i(t), i = 1, 2, 4, 5$, are the given $2\pi/\sigma$-periodic functions,

$$\eta_i(t) = \eta_{ia}^{(0)} + \sum_{k=1}^{\infty}\Big[\eta_{ia}^{(k)}\cos(k\sigma t) + \mu_{ia}^{(k)}\sin(k\sigma t)\Big], \quad \eta_{ia}^{(k)} \sim \mu_{ia}^{(k)} = O(\epsilon), \quad (13)$$

  where σ is the circular forcing frequency; the lowest-order harmonic component should not be zero, i.e.

$$\sum_{i=1,2,4,5} |\eta_{ia}^{(1)}| + |\mu_{ia}^{(1)}| \neq 0; \quad (14)$$

- the forcing frequency σ is close to the lowest natural sloshing frequency σ_{11} so that the so-called Moiseev detuning condition

$$\bar{\sigma}_{11}^2 - 1 = O(\epsilon^{2/3}), \quad \bar{\sigma}_{11} = \sigma_{11}/\sigma \quad (15)$$

  is satisfied;

– there are no resonance amplifications of $p_{mj}, r_{mj}, \; m\, j \neq 1$ that implies

$$\begin{aligned} & m - \bar{\sigma}_{1k} \geq O(1), \quad \bar{\sigma}_{mi} = \sigma_{mi}/\sigma, \quad m,k \geq 2; \\ & \bar{\sigma}^2_{0i} - 4 \sim \bar{\sigma}^2_{2i} - 4 \sim \bar{\sigma}^2_{3i} - 9 \sim \bar{\sigma}^2_{1(i+1)} - 9 \geq O(1), \quad i \geq 1; \end{aligned} \tag{16}$$

the second row means that there is no the so-called secondary resonance [2].

Our goal consists of constructing an asymptotic periodic solution of (9)–(12) and (13). The right-hand sides of (9) are

$$\sigma^2 P_1 \kappa_{11} \sum_{k=1}^{\infty} \Big[(k\eta^{(k)}_{1a} - (kS_1 - g/\sigma^2)\eta^{(k)}_{5a}) \cos(k\sigma t) \\ + (k\mu^{(k)}_{1a} - (kS_1 - g/\sigma^2)\mu^{(k)}_{5a}) \sin(k\sigma t) \Big],$$

$$\sigma^2 P_1 \kappa_{11} \sum_{k=1}^{\infty} \Big[(k\eta^{(k)}_{2a} + (kS_1 - g/\sigma^2)\eta^{(k)}_{4a}) \cos(k\sigma t) \\ + (k\mu^{(k)}_{2a} + (kS_1 - g/\sigma^2)\mu^{(k)}_{4a}) \sin(k\sigma t) \Big].$$

Because of (15), neglecting the higher-order terms, $o(\epsilon)$, allows for replacing $g/\sigma^2 \to g/\sigma^2_{11}$ and, therefore, amplitudes of the first Fourier harmonics are

$$\begin{aligned} \epsilon_x &= P_1 \kappa_{11} (\eta^{(1)}_{1a} - (S_1 - g/\sigma^2_{11})\eta^{(1)}_{5a}), \\ \bar{\epsilon}_x &= P_1 \kappa_{11} (\mu^{(1)}_{1a} - (S_1 - g/\sigma^2_{11})\mu^{(1)}_{5a}), \\ \bar{\epsilon}_y &= P_1 \kappa_{11} (\eta^{(1)}_{2a} + (S_1 - g/\sigma^2_{11})\eta^{(1)}_{4a}), \\ \epsilon_y &= P_1 \kappa_{11} (\mu^{(1)}_{2a} + (S_1 - g/\sigma^2_{11})\mu^{(1)}_{4a}). \end{aligned} \tag{17}$$

Here, ϵ_x and $\bar{\epsilon}_x$ appear in the front of $\cos \sigma t$ and $\sin \sigma t$ and imply the forcing components in the Ox direction, but $\bar{\epsilon}_y$ and ϵ_y correspond to the $\cos \sigma t$ and $\sin \sigma t$ harmonics along the Oy axis. Because of (14), rotating the Oxy frame around Oz can always help getting the non-zero first-harmonic forcing component along Ox, i.e. $\epsilon^2_x + \bar{\epsilon}^2_x \neq 0$. Furthermore, the periodicity condition is defined within to an arbitrary phase shift and one can assume, without loss of generality, that

$$\epsilon_x > 0, \quad \bar{\epsilon}_x = 0. \tag{18}$$

Henceforth, we follow the Bubnov-Galerking procedure [2] by posing the lowest-order asymptotic solution component

$$p_{11}(t) = a\cos(\sigma t) + \bar{a}\sin(\sigma t) + O(\epsilon), \; r_{11}(t) = \bar{b}\cos(\sigma t) + b\sin(\sigma t) + O(\epsilon), \tag{19}$$

where $a, \bar{a}, \bar{b}$, and b are of $O(\epsilon^{1/3})$. The second- and third-order generalised coordinates can be found from (10) and (11), (12), respectively. This gives, in particular,

$$p_{0k}(t) = s_{0k}(a^2 + \bar{a}^2 + b^2 + \bar{b}^2) \\ + s_{1k} \left[(a^2 - \bar{a}^2 - b^2 + \bar{b}^2)\cos(2\sigma t) + 2(a\bar{a} + b\bar{b})\sin(2\sigma t) \right] + o(\epsilon), \tag{20a}$$

$$p_{2k}(t) = c_{0k}(a^2 + \bar{a}^2 - b^2 - \bar{b}^2) \\ + c_{1k}\left[(a^2 - \bar{a}^2 + b^2 - \bar{b}^2)\cos(2\sigma t) + 2(a\bar{a} - b\bar{b})\sin(2\sigma t)\right] + o(\epsilon), \quad \text{(20b)}$$

$$r_{2k}(t) = 2c_{0k}(a\bar{b} + b\bar{a}) \\ + 2c_{1k}\left[(a\bar{b} - b\bar{a})\cos(2\sigma t) + (ab + \bar{a}\bar{b})\sin(2\sigma t)\right] + o(\epsilon), \quad \text{(20c)}$$

where

$$s_{0k} = \tfrac{1}{2}\left(\frac{d_{10,k} - d_{8,k}}{\bar{\sigma}_{0k}^2}\right), \quad s_{1k} = \frac{d_{10,k} + d_{8,k}}{2(\bar{\sigma}_{0k}^2 - 4)}, \quad \bar{\sigma}_{0k} = \frac{\sigma_{0k}}{\sigma}, \\ c_{0k} = \tfrac{1}{2}\left(\frac{d_{9,k} - d_{7,k}}{\bar{\sigma}_{2k}^2}\right), \quad c_{1k} = \frac{d_{9,k} + d_{7,k}}{2(\bar{\sigma}_{2k}^2 - 4)}, \quad \bar{\sigma}_{2k} = \frac{\sigma_{2k}}{\sigma}. \quad \text{(21)}$$

Substituting (19) and (20) into (9) and gathering the first harmonic terms, $\cos\sigma t$ and $\sin\sigma t$, lead to the solvability (secular) equations

$$\begin{cases} ① : \ a\left[(\bar{\sigma}_{11}^2 - 1) + m_1(a^2 + \bar{a}^2 + \bar{b}^2) + m_3 b^2\right] + (m_1 - m_3)\bar{a}\bar{b}b = \epsilon_x, \\ ② : \ b\left[(\bar{\sigma}_{11}^2 - 1) + m_1(b^2 + \bar{b}^2 + \bar{a}^2) + m_3 a^2\right] + (m_1 - m_3)\bar{a}a\bar{b} = \epsilon_y, \\ ③ : \ \bar{a}\left[(\bar{\sigma}_{11}^2 - 1) + m_1(a^2 + \bar{a}^2 + b^2) + m_3 \bar{b}^2\right] + (m_1 - m_3)a\bar{b}b = 0, \\ ④ : \ \bar{b}\left[(\bar{\sigma}_{11}^2 - 1) + m_1(b^2 + \bar{b}^2 + a^2) + m_3 \bar{a}^2\right] + (m_1 - m_3)\bar{a}ab = \bar{\epsilon}_y \end{cases} \quad \text{(22)}$$

with respect to $a, \bar{a}, \bar{b}$ and b. The coefficients m_1 and m_3 are computed by the formulas

$$m_1 = -\tfrac{1}{2}d_1 + \sum_{j=1}^{I_r}\Big[c_{1j}\left(\tfrac{1}{2}d_3^{(j)} - 2d_4^{(j)}\right) \\ + s_{1j}\left(\tfrac{1}{2}d_5^{(j)} - 2d_6^{(j)}\right) - s_{0j}d_5^{(j)} - c_{0j}d_3^{(j)}\Big], \quad \text{(23a)}$$

$$m_3 = \tfrac{1}{2}d_1 - 2d_2 + \sum_{j=1}^{I_r}\Big[c_{1j}\left(\tfrac{3}{2}d_3^{(j)} - 6d_4^{(j)}\right) \\ + s_{1j}\left(-\tfrac{1}{2}d_5^{(j)} + 2d_6^{(j)}\right) - s_{0j}d_5^{(j)} + c_{0j}d_3^{(j)}\Big]. \quad \text{(23b)}$$

After finding $a, \bar{a}, \bar{b}$ and b from (22), the second- and third-order components of the asymptotic solution are fully determined. Coefficients in this solution as well as m_1 and m_3 in (22) are functions of h, r_1 and the forcing frequency σ. Utilising (15) shows that the latter dependence can be neglected by substituting $\sigma = \sigma_{11}$ into the corresponding expressions. Dependence on σ *remains only* in the $(\bar{\sigma}_{11}^2 - 1)$-quantity of (22).

Calculations show that (16) is fulfilled for fairy deep liquid depths, $1.2 \lesssim h$, and the conditions

$$O(1) = m_1 < 0 \text{ and } O(1) = m_1 + m_3 > 0 \quad \text{(24)}$$

are satisfied. This means, in particular, that $m_3 > 0$ and $m_1 \neq m_3$.

One can follow [2] to study the stability of the asymptotic solution by using the linear stability analysis and the multi-timing technique. For this purpose, we introduce the slowly varying time $\tau = \epsilon^{2/3}\sigma t/2$ (the order $\epsilon^{2/3}$ is chosen to match the lowest asymptotic terms in the multi-timing technique), the Moiseev detuning (15), and express the infinitesimally perturbed solution

$$p_{11} = (a + \alpha(\tau)) \cos \sigma t + (\bar{a} + \bar{\alpha}(\tau)) \sin \sigma t + o(\epsilon^{1/3}),$$
$$r_{11} = (\bar{b} + \bar{\beta}(\tau)) \cos \sigma t + (b + \beta(\tau)) \sin \sigma t + o(\epsilon^{1/3}), \tag{25}$$

where $a, \bar{a}, b$ and $\bar{b}$ are known and $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ are their relative perturbations depending on τ. Inserting (25) into the Narimanov-Moiseev modal equations, gathering terms of the lowest asymptotic order and keeping linear terms in $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ lead to the following linear system of ordinary differential equations

$$\frac{d\mathbf{c}}{d\tau} + \mathcal{C}\mathbf{c} = 0, \tag{26}$$

where $\mathbf{c} = (\alpha, \bar{\alpha}, \beta, \bar{\beta})^T$ and the matrix $\mathcal{C}$ has the elements

$$c_{11} = -[2a\bar{a}\, m_1 + (m_1 - m_3)\, b\bar{b}];$$
$$c_{12} = -[(\bar{\sigma}_{11}^2 - 1) + m_1(a^2 + 3\bar{a}^2 + b^2) + m_3\, \bar{b}^2],$$
$$c_{13} = -[2\bar{a}b\, m_1 + (m_1 - m_3)\, a\bar{b}];\; c_{14} = -[2\bar{a}\bar{b}\, m_3 + (m_1 - m_3)\, ab],$$
$$c_{21} = (\bar{\sigma}_{11}^2 - 1) + m_1(3a^2 + \bar{a}^2 + \bar{b}^2) + m_3\, b^2;\; c_{22} = 2a\bar{a}\, m_1 + (m_1 - m_3)\, b\bar{b},$$
$$c_{23} = 2ab\, m_3 + (m_1 - m_3)\, \bar{a}\bar{b};\; c_{24} = 2a\bar{b}\, m_1 + (m_1 - m_3)\, \bar{a}b,$$
$$c_{31} = 2m_1\, a\bar{b} + (m_1 - m_3)\, b\bar{a};\; c_{32} = 2m_3\, \bar{a}\bar{b} + (m_1 - m_3)\, ab,$$
$$c_{33} = 2m_1\, b\bar{b} + (m_1 - m_3)\, a\bar{a};\; c_{34} = (\bar{\sigma}_{11}^2 - 1) + m_1(b^2 + 3\bar{b}^2 + a^2) + m_3\, \bar{a}^2,$$
$$c_{41} = -[2m_3\, ab + (m_1 - m_3)\, \bar{a}\bar{b}];\; c_{42} = -[2\bar{a}b\, m_1 + (m_1 - m_3)\, a\bar{b}],$$
$$c_{43} = -[(\bar{\sigma}_{11}^2 - 1) + m_1(3b^2 + \bar{b}^2 + \bar{a}^2) + m_3\, a^2];$$
$$c_{44} = -[2b\bar{b}\, m_1 + (m_1 - m_3)\, a\bar{a}].$$

The instability of the asymptotic solution can be evaluated by studying (26). Its fundamental solution depends on the eigenvalue problem $\det[\lambda E + C] = 0$, where E is the identity matrix. Computations give the following characteristic polynomial

$$\lambda^4 + c_1\lambda^2 + c_0 = 0, \tag{27}$$

where c_0 is the determinant of $\mathcal{C}$ and c_1 is a complicated function of the elements of $\mathcal{C}$. As discussed in [2], the stability requires $c_0 > 0, c_1 > 0$ and $c_1^2 - 4c_0 > 0$. When at least one of the inequalities is not fulfilled, the steady-state wave regime associated with the dominant amplitudes $a, \bar{a}, b$ and $\bar{b}$ is not stable.

4. Classification of steady-state (periodic) solutions

The steady-state (periodic) sloshing can be classified by analysing the lowest-order component (19) which gives the dominant wave contribution. The lowest-order amplitudes $a, \bar{a}, b$ and $\bar{b}$ follow from the secular system (22) which does not involve the super-harmonic components from (13). This means that the resonant sloshing regimes are, within to the higher-order terms, the same as if the tank performs the *artificial horizontal harmonic* motions

$$
\begin{aligned}
&(\kappa_{11}P_1)\,\eta_1(t) = \epsilon_x \cos\sigma t, \\
&(\kappa_{11}P_1)\,\eta_2(t) = \bar{\epsilon}_y \cos\sigma t + \epsilon_y \sin\sigma t, \quad \eta_4(t) = \eta_5(t) = 0
\end{aligned} \tag{28}
$$

that define, by accounting for (18), either longitudinal ($\epsilon_y = 0$) or *elliptic (rotary)* ($\epsilon_y \neq 0$) harmonic tank motion. The latter occurs along the trajectory

$$
\frac{\epsilon_y^2 + \bar{\epsilon}_y^2}{\epsilon_x^2} x^2 + y^2 - 2\frac{\bar{\epsilon}_y}{\epsilon_x} xy = \epsilon_y^2. \tag{29}
$$

For the longitudinal tank motions ($\epsilon_y = 0$), one can rotate the Oxy frame around Oz to get the artificial tank vibrations by (28) occurring along the Ox axis. The forcing amplitudes become then $\epsilon_x > 0$ and $\bar{\epsilon}_y = \epsilon_y = 0$ and the secular system (22) has only two analytical solutions well known from, for example, [4]. The first solution implies the so-called *planar* steady-state wave ($\bar{a} = \bar{b} = b = 0$). The nonzero lowest-order amplitude parameter a is governed by

$$
a\left[(\bar{\sigma}_{11}^2 - 1) + m_1 a^2\right] = \epsilon_x. \tag{30}
$$

This solution is characterised by the zero transverse wave component, namely, $r_{mi}(t) \equiv 0$. The second solution corresponds to *swirling* whose longitudinal ($a \neq 0$) and transverse ($b \neq 0$) amplitude parameters come from the system

$$
\begin{cases}
a\left[(\bar{\sigma}_{11}^2 - 1) + (m_1 + m_3)a^2\right] = \dfrac{m_1}{m_1 - m_3}\epsilon_x, \\
b^2 = -\dfrac{(\bar{\sigma}_{11}^2 - 1) + m_3 a^2}{m_1} > 0.
\end{cases} \tag{31}
$$

Why the solution $\bar{a} = \bar{b} = 0$, $ab \neq 0$ is called swirling is discussed in [4].

When the artificial horizontal harmonic motions occur along an elliptic trajectory ($\epsilon_y \neq 0$), rotating the Oxy frame around Oz helps superposing Ox with the major axis of the ellipse. This new position of the Oxy frame implies that

$$
\bar{\epsilon}_y = 0, \quad 0 < \epsilon_y \leq \epsilon_x \neq 0 \tag{32}
$$

in (22). The following equalities

$$
\begin{aligned}
\bar{a}\cdot ① - a\cdot ③ &= \bar{b}\cdot ② - b\cdot ④ \\
&= (m_1 - m_3)[a\bar{a}(\bar{b}^2 - b^2) + \bar{b}b(\bar{a}^2 - a^2)] = \bar{a}\epsilon_x = \bar{b}\epsilon_y,
\end{aligned} \tag{33a}
$$

$$
\begin{aligned}
\bar{b}\cdot ① - a\cdot ④ &= \bar{a}\cdot ② - b\cdot ③ \\
&= (m_1 - m_3)[b\bar{a}(\bar{b}^2 - a^2) + \bar{b}a(\bar{a}^2 - b^2)] = \bar{b}\epsilon_x = \bar{a}\epsilon_y,
\end{aligned} \tag{33b}
$$

$$
b\cdot ① - a\cdot ② = (m_1 - m_3)(a^2 - b^2)(ab - \bar{a}\bar{b}) = b\epsilon_x - a\epsilon_y \tag{33c}
$$

can then be treated as solvability conditions of (22).

When $0 < \epsilon_y < \epsilon_x$, the homogeneous linear system (33a)–(33b) with respect to $\bar{a}$ and $\bar{b}$ has only trivial solution $\bar{a} = \bar{b} = 0$. Equation (33c) shows then that $a\,b \neq 0$ (and $a \neq b$) and, therefore, the only nonzero amplitudes a and b

always determine *swirling.* The amplitudes are governed by (22) which can be rewritten in the equivalent form

$$\begin{cases} b\left[(m_1-m_3)b^2+\left(\dfrac{\epsilon_x}{a}-(m_1-m_3)a^2\right)\right]=\epsilon_y, \\ (\bar{\sigma}_{11}^2-1)=\dfrac{\epsilon_x}{a}-m_1a^2-m_3b^2, \quad a\neq 0. \end{cases} \tag{34}$$

The first equality is a depressed cubic with respect to b whose coefficient at the linear term is a function of a. The second equality computes the forcing frequency, σ/σ_{11} $(\bar{\sigma}_{11}^2-1)$, as a function of a and b. A numerical procedure may suggest varying a in an admissible range, solving the depressed cubic (finding $b=b(a)$), and computing σ/σ_{11} as a function a and $b=b(a)$. When solving the depressed cubic, one should check for the discriminant

$$\Delta=-4\left(\frac{\epsilon_x}{a\,(m_1-m_3)}-a^2\right)^3-27\left(\frac{\epsilon_y}{m_1-m_3}\right)^2, \quad 0<\epsilon_y<\epsilon_x. \tag{35}$$

Cartano's theorem says that, (i) if $\Delta>0$, then there are three distinct real roots for b, (ii) if $\Delta=0$, then the equation has at least one multiple root and all its roots are real, and (iii) if $\Delta<0$, then the equation has one real root and two nonreal complex conjugate roots.

When considering Δ as a function of a, a simple analysis shows that, if $m_1-m_3<0$, there exists only a negative real root $a_*<0$ of $\Delta(a_*)=0$ so that $\Delta(a)>0$ for $a<a_*$ and $0<a$ (three real solutions) but $\Delta(a)<0$ for $a_*<a<0$ (a unique real solution). Analogously, if $m_1-m_3>0$, there exists only a positive real root $a_*>0$ of $\Delta(a_*)=0$ so that $\Delta(a)>0$ for $a<0$ and $a_*<a$ but $\Delta(a)<0$ for $0<a<a_*$.

When $\bar{\epsilon}_y=0$, $\epsilon_y=\epsilon_x\neq 0$ (artificial rotary harmonic motions of the tank), equations (33a) and (33b) are unable to derive that $\bar{a}$ and $\bar{b}$ are zeros but deduce, instead, $\bar{a}=\bar{b}=c$. The latter makes ③ ≡ ④ in (22). By taking the sum ① + ② and the difference ① − ②, we transform (22) to the form

$$\begin{cases} (a+b)\big\{(\bar{\sigma}_{11}^2-1)+m_1(a^2+b^2)+ \\ \qquad\qquad +(3m_1-m_3)c^2-(m_1-m_3)ab\big\}=2\epsilon, \\ (a-b)\big[(\bar{\sigma}_{11}^2-1)+m_1(a^2+b^2)+ \\ \qquad\qquad +(m_1+m_3)c^2+(m_1-m_3)ab\big]=0, \\ c\left[(\bar{\sigma}_{11}^2-1)+m_1(a^2+b^2)+(m_1+m_3)c^2+(m_1-m_3)ab\right]=0, \end{cases} \tag{36}$$

in which the two homogeneous equations contain identical expressions in the square bracket. These expressions are multiplied by $(a-b)$ and c, respectively.

We adopt $a_+=\frac{1}{2}(a+b), a_-=\frac{1}{2}(a-b)$ instead of a and b. When both a_- and c are zeros, we arrive at

$$\bar{a}=\bar{b}=0, \quad a_+=a=b, \quad a_+\left[(\bar{\sigma}_{11}^2-1)+(m_1+m_3)a_+^2\right]=\epsilon \tag{37}$$

which imply *rotary* (circular swirling) waves characterised by equal longitudinal (along Ox) and transverse (along Oy) amplitude components, $p_{11}(t)=a_+\cos(\sigma t)+O(\epsilon)$, $r_{11}(t)=a_+\sin(\sigma t)+O(\epsilon)$. The rotary waves are co-directed with the rotary tank motion.

When either $a_- \neq 0$ or $c \neq 0$, the square bracket expression of (36) must be zero. This makes the second and third equalities of (36) automatically satisfied and, therefore, three amplitude parameters a_+, a_- and c should be found from the two equalities

$$\begin{cases} a_+ \left[(\bar{\sigma}_{11}^2 - 1) + 4m_1 a_+^2\right] = -\dfrac{m_1 + m_3}{2(m_1 - m_3)}\,\epsilon, \\ a_-^2 + c^2 = -\dfrac{(\bar{\sigma}_{11}^2 - 1) + (3m_1 - m_3)a_+^2}{(m_1 + m_3)} > 0, \end{cases} \tag{38}$$

which define the following lowest-order steady-state solution component

$$\begin{aligned} p_{11}(t) &= (a_+ + a_-)\cos(\sigma t) + c\sin(\sigma t) + O(\epsilon), \\ r_{11}(t) &= (a_+ - a_-)\sin(\sigma t) + c\cos(\sigma t) + O(\epsilon). \end{aligned} \tag{39}$$

The amplitude a_+ can be found from the first equation of (38) but the amplitudes a_- and c are not uniquely defined. Only the sum $a_-^2 + c^2$ can be found for any fixed pair $(\bar{\sigma}_{11}^2, a_+)$ from the first cubic equation. This defines a manifold $a_+ = a_+(\sigma/\sigma_{11}), a_-^2 + c^2 = F(\sigma/\sigma_{11}, a_+)$ in the four-dimensional space $(\sigma/\sigma_{11}, a_+, a_-, c)$. Numerical analysis of the solution (39) shows that it is *unstable on the aforementioned manifold* due to $c_0 = 0$ in the characteristic equation (27).

When $c = 0$, system (38) defines the three-dimensional response curves $a_+ = a_+(\sigma/\sigma_{11})$, $a_- = a_-(\sigma/\sigma_{11})$ which implies the solution

$$p_{11}(t) = (a_+ + a_-)\cos(\sigma t) + O(\epsilon), \quad r_{11}(t) = (a_+ - a_-)\sin(\sigma t) + O(\epsilon) \tag{40}$$

which has the same form as for the elliptically-excited swirling with $\epsilon_y < \epsilon_x$.

5. Conclusions

By using the Narimanov-Moiseev type modal theory [7], the steady-state (periodic) resonant waves in an upright circular cylindrical tank with a fairly deep liquid depth are analysed when the tank performs an arbitrary small-magnitude sway-surge-pitch-roll periodic motion. The forcing frequency is close to the lowest natural sloshing frequency. The analysis shows that, within to the higher-order terms, the resonant sloshing is the same as that due to either longitudinal or elliptic/rotary horizontal harmonic tank motions. The longitudinal case is well known from the literature. Planar (in the excitation plane) and swirling waves were established and described. In the present paper, the cases of elliptic and rotary excitations are studied to show that they always lead to swirling, which can be either co- or counter-directed with respect to the forcing direction. The co-directed wave converts then to the rotary wave regime when the elliptic forcing tends to the rotary one. The effective frequency range of the stable counter-directed swirling becomes unstable in this limit case.

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THE BEST M-TERM TRIGONOMETRIC APPROXIMATIONS OF CLASSES OF (ψ, β)-DIFFERENTIABLE PERIODIC MULTIVARIATE FUNCTIONS IN THE SPACE $L^{\psi}_{\beta,1}$

K. V. Shvai

Резюме. Встановлено порядкові оцінки найкращих M–членних тригонометричних наближень періодичних функцій D^{ψ}_{β} у просторі L_q, $1 < q \leq 2$. Використовуючи одержані результати, встановлено порядкові співвідношення цих величин для класів $L^{\psi}_{\beta,1}$.

Abstract. Obtained here are the order estimates of the best M–term trigonometric approximations of periodic functions D^{ψ}_{β} in the space $L_q, 1 < q \leq 2$. The results are applied to establish the order estimates of the same quantities for classes $L^{\psi}_{\beta,1}$.

1. Introduction

Let us introduce all necessary denotations and give a definition of the approximative characteristic to investigate.

Let $L_q(\pi_d)$, $1 \leq q \leq \infty$, — be the space of functions f, 2π–periodic by each variable, with the finite norm

$$\|f\|_{L_q(\pi_d)} = \|f\|_q = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^q dx\right)^{\frac{1}{q}},\ 1 \leq q < \infty,$$

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|,$$

where $x = (x_1, \ldots, x_d)$ is the element of Euclidean space $\mathbb{R}^d$, $d \geq 1$, and $\pi_d = \prod\limits_{j=1}^{d} [-\pi, \pi]$. Suppose further that for the functions $f \in L_q(\pi_d)$ the condition

$$\int\limits_{-\pi}^{\pi} f(x) dx_j = 0,\ j = \overline{1, d},$$

holds.

Let us consider the Fourier series for the function $f \in L_1(\pi_d)$

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{i(k,x)},$$

Key words. The best trigonometric approximations; Bernoulli kernel; order estimates; Fourier series.

where

$$\widehat{f}(k) = (2\pi)^{-d} \int\limits_{\pi_d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f, $(k, x) = k_1 x_1 + \ldots + k_d x_d$.

Let $\psi_j(\cdot) \neq 0$ be arbitrary functions of the natural argument, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$. Assume that the series

$$\sum_{k \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^{d} \frac{e^{i\frac{\pi\beta_j}{2}\operatorname{sgn} k_j}}{\psi_j(|k_j|)} \widehat{f}(k) e^{i(k,x)},$$

where $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$, are the Fourier series of some summable on π_d function. Following O. I. Stepanets [1, c. 25], (see also [2, c. 132]), let us call it (ψ, β)–derivative of the function f and denote it as f^{ψ}_{β}. A set of functions f, for which (ψ, β)–derivatives exist, is denoted as L^{ψ}_{β}.

If the condition $\left\| f^{\psi}_{\beta}(\cdot) \right\|_p \leq 1,\ 1 \leq p \leq \infty$, holds then $f \in L^{\psi}_{\beta,p}$.

The article deals with the best M–term trigonometric approximations of the functions D^{ψ}_{β} whose Fourier series are written in a form

$$\sum_{k \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^{d} \psi_j\left(|k_j|\right) e^{i\frac{\pi\beta_j}{2}\operatorname{sgn} k_j} e^{i(k,x)}.$$

Note that if $\psi_j\left(|k_j|\right) = |k_j|^{-r_j}$, $r_j > 0$, $k_j \in \mathbb{Z}\backslash\{0\}$, $j = \overline{1, d}$, D^{ψ}_{β} is a multivariate analogue of the Bernoulli kernel (see, e.g., [3, c. 31]).

Each of the functions $f \in L^{\psi}_{\beta,p}$ can be presented in a form of convolution

$$f(x) = \left(\varphi * D^{\psi}_{\beta}\right)(x) = (2\pi)^{-d} \int\limits_{\pi_d} \varphi(x - t) D^{\psi}_{\beta}(t) dt, \tag{1}$$

where $\|\varphi\|_p \leq 1$, and the function $\varphi(\cdot)$ almost everywhere coincides with f^{ψ}_{β}.

As an apparatus of the approximation we will use trigonometric polynomials of the form

$$P\left(\theta_M; x\right) = \sum_{k \in \theta_M} c_k e^{i(k,x)},$$

where θ_M is an arbitrary set of M different vectors $k = (k_1, \ldots k_d)$ and $c_k \in \mathbb{C}$. For $f \in L_q\left(\pi_d\right)$, $1 \leq q \leq \infty$, the quantity

$$e_M(f)_q = \inf_{\theta_M} \inf_{P(\theta_M;\cdot)} \left\| f(\cdot) - P\left(\theta_M; \cdot\right) \right\|_q. \tag{2}$$

is called the best M–term trigonometric approximation of the function f. And the quantity

$$e^{\perp}_M(f)_q = \inf_{\theta_M} \left\| f(\cdot) - \sum_{k \in \theta_M} \widehat{f}(k) e^{i(k,x)} \right\|_q, \tag{3}$$

is called the best orthogonal trigonometric approximation of the function f. It is obvious that the relation

$$e_M(f)_q \le e^{\perp}_M(f)_q,\ 1 \le q \le \infty, \tag{4}$$

holds. If $F \subset L_q$ is some functional class then denote

$$e_M(F)_q = \sup_{f\in F} e_M(f)_q, \tag{5}$$

and, accordingly,

$$e^{\perp}_M(F)_q = \sup_{f\in F} e^{\perp}_M(f)_q. \tag{6}$$

The quantity (2) appeared at first in the paper of S. B. Stechkin [4] in formulating an absolute convergence criterion for orthogonal series. Later the quantity (5) for classes of periodic functions of one and many variables was investigated in the papers of V. N. Temlyakov [3], [5–7], E. S. Belinskii [8–10,12], A. S. Romanyuk [13–20], A. S. Fedorenko [21–23], N. M. Konsevych [24, 25], V. V. Shkapa [26] and others.

The quantities (3) and (6) were considered by E. S. Belinskii (see, e.g., [12]), and later their exploration was further developed in the works of many authors. The detailed bibliography can be found in [20, 27].

The results of the article are formulated in order-relation terms. So, further for the quantities A and B under the notation $A \ll B$ we will understand the existance of a positive constant C_1 such that $A \le C_1 B$. If the conditions $A \ll B$ and $B \ll A$ hold then we write $A \asymp B$. All constants in order relations can depend only on the parameters that are in the definitions of class and metric in which the approximation is carried out, and on the dimension of the space $\mathbb{R}^d$.

2. Auxiliary statements

To formulate and prove the results of the article some notations and auxiliary statements will be needed.

Let D be a set of functions $\psi(\cdot)$ of natural argument that satisfy the conditions

1) $\psi(\cdot)$ are positive and nonincreasing;

2) $\exists M > 0$ such that $\forall l \in \mathbb{N}$ $\frac{\psi(l)}{\psi(2l)} \le M$.

Note that to the indicated set of functions belong, in particular, functions $\psi(|k|) = |k|^{-r}$, $\psi(|k|) = |k|^{-r} \ln^{\alpha}(|k|+1)$, $r > 0$, $k \in \mathbb{Z}\backslash\{0\}$, $\alpha \in \mathbb{R}$ and others.

Further, let us put into conformity to each vector $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N} \cup \{0\}$, $j = \overline{1, d}$, a set

$$\rho(s) = \left\{k = (k_1, \dots, k_d) :\ \left[2^{s_j-1}\right] \le |k_j| < 2^{s_j},\ j = \overline{1, d}\right\},$$

where $[\cdot]$ is the whole part, and for $f \in L_1(\pi_d)$ put

$$\delta_s(f, x) = \sum_{k\in\rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where $\widehat{f}(k)$ are the Fourier coefficients of this function. Note that the unifications of "blocks" $\rho(s)$, $(s,1)=s_1+\ldots+s_d<n$, $n\in\mathbb{N}$, form a set Q_n that is called "step-hyperbolic cross" [3, c. 7]. The quantity of points in this set is of the order $2^n n^{d-1}$ [3, c. 70].

The following propositions hold.

Proposition 7. *[27] Let $1<q<\infty$, $\psi_j\in D$, $\beta_j\in\mathbb{R}$, $j=\overline{1,d}$, and, besides, there exists $\varepsilon>0$ such that $\psi_j(|k_j|)\,|k_j|^{1-\frac{1}{q}+\varepsilon}$ are nonincreasing. Then for all natural M and n that satisfy the condition $M\asymp 2^n n^{d-1}$, the following relations hold*

$$\Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}\ll e_M^{\perp}\left(D_{\beta}^{\psi}\right)_q\ll$$

$$\ll\Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)},$$

$$\Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}\ll e_M^{\perp}\left(L_{\beta,1}^{\psi}\right)_q\ll$$

$$\ll\Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}.$$

where

$$\Phi(n)=\min_{(s,1)=n}\prod_{j=1}^{d}\psi_j\left(2^{s_j}\right),\quad \Psi(n)=\max_{(s,1)=n}\prod_{j=1}^{d}\psi_j\left(2^{s_j}\right).$$

Proposition 8. *[3, c. 28] For an arbitrary function $f\in L_q(\pi_d)$, $1<q<p\le\infty$, holds*

$$\|f\|_q^q\gg\sum_s\|\delta_s(f,\cdot)\|_p^q\cdot 2^{(s,1)\left(\frac{1}{p}-\frac{1}{q}\right)q}.$$

To make further speculations we need one more relation which follows from a more general result of S. N. Nikolskii (see, e.g., [28, c. 25]).

Proposition 9. *For all functions $f\in L_q(\pi_d), 1\le q<\infty$, holds*

$$e_M(f)_q=\inf_{\theta_M}\sup_{\substack{P\in L^{\perp}(\theta_M),\\ \|P\|_{q'}\le 1}}\left|\int\limits_{\pi_d}f(x)\overline{P(x)}dx\right|,$$

where $L^{\perp}(\theta_M)$ is a set of functions that is orthogonal to the subset of trigonometric polynomials with the numbers of harmonics from the set θ_M, and $\frac{1}{q}+\frac{1}{q'}=1$.

3. The best M-term trigonometric approximations

The following statement holds.

Theorem 1. *Let $1<q\le 2$, $\psi_j\in D$, $\beta_j\in\mathbb{R}$, $j=\overline{1,d}$, and, besides, there exists $\varepsilon>0$ such that $\psi_j(|k_j|)\,|k_j|^{1-\frac{1}{q}+\varepsilon}$ are nonincreasing. Then for arbitrary natural M and n that satisfy condition $M\asymp 2^n n^{d-1}$, we have the estimate*

$$\Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}\ll e_M\left(D_{\beta}^{\psi}\right)_q\ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \qquad (7)$$

where

$$\Phi(n) = \min_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right), \quad \Psi(n) = \max_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right).$$

Proof. The upper estimate follows from (4) and proposition 7, that is

$$e_M\left(D_\beta^\psi\right)_q \le e_M^\perp\left(D_\beta^\psi\right)_q \ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \ 1 < q \le 2. \qquad (8)$$

Let us go to the establishment of the lower estimate in (7). For the given M let us choose n so that the relation $M \asymp 2^n n^{d-1}$ holds. Note that the consideration of the case $\beta = 0$ is sufficient to receive a corresponding estimate.

Let

$$D^\psi(x) = D_0^\psi(x) = 2^d \sum_s \sum_{k\in\rho^+(s)} \prod_{j=1}^{d} \psi_j\left(k_j\right) \cos k_j x_j,$$

where $\rho^+(s) = \{k = (k_1, \ldots, k_d) : \ \left[2^{s_j-1}\right] \le k_j < 2^{s_j}, \ j = \overline{1,d}\}$. By S we denote a set of vectors $s \in \mathbb{N}^d$, such that $(s,1) = n$ and $|\theta_M \cap \rho^+(s)| \le \frac{1}{2}|\rho^+(s)|$ hold. Then, using proposition 8 (if $p = 2$), we get

$$I_1 = \left\|D_\beta^\psi(\cdot) - P\left(\theta_M;\cdot\right)\right\|_q \gg$$

$$\gg \left(\sum_s \left\|\delta_s\left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q \cdot 2^{(s,1)\left(\frac{1}{2}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{(s,1)=n} \left\|\delta_s\left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left\|\sum_{k\in\rho^+(s)} \left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q\right)^{\frac{1}{q}}.$$

Further, according to the Parseval equality, we can write

$$I_1 \gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left(\sum_{k\in\rho^+(s)} \left(\prod_{j=1}^{d} \psi_j\left(k_j\right)\right)^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left(\min_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right)\right)^q 2^{(s,1)\frac{q}{2}}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \Phi(n) \left(2^{\frac{nq}{2}}|S|\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)}\Phi(n)2^{\frac{n}{2}}n^{\frac{d-1}{q}} = \Phi(n)2^{n\left(1-\frac{1}{q}\right)}n^{\frac{d-1}{q}} \tag{9}$$

So, taking into account that $M \asymp 2^n n^{d-1}$, from (9) we receive

$$e_M\left(D^{\psi}_{\beta}\right)_q \gg \Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)},\ 1 < q \leq 2. \tag{10}$$

The lower estimate is proven. The relation (7) follows from (8) and (10).

The theorem is proven.

Remark 2. *In the case* $\psi_j(|k_j|) = |k_j|^{-r_j}$, $r_j > 1 - \frac{1}{q}$, $1 \leq q \leq 2$, $k_j \in \mathbb{Z}\backslash\{0\}$, $j = \overline{1, d}$, *corresponding results were obtained by E. S. Belinskii [8, 9].*

Further, by using the lower estimate established in theorem 1, we get estimates of the best M–term trigonometric approximations for the classes of functions $L^{\psi}_{\beta,1}$.

The theorem holds.

Theorem 2. *Let* $1 < q \leq 2$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$, *and, besides, there exists* $\varepsilon > 0$ *such that* $\psi_j(|k_j|)|k_j|^{1-\frac{1}{q}+\varepsilon}$ *are nonincreasing. Then for arbitrary natural* M *and* n *that satisfy condition* $M \asymp 2^n n^{d-1}$, *the relation holds*

$$\Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)} \ll e_M\left(L^{\psi}_{\beta,1}\right)_q \ll$$
$$\ll \Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}. \tag{11}$$

Proof. The upper estimate follows from the relation (4) and the already known result for the best orthogonal trigonometric approximations. Given proposition 7 we get

$$e_M\left(L^{\psi}_{\beta,1}\right)_q \ll e^{\perp}_M\left(L^{\psi}_{\beta,1}\right)_q \ll$$
$$\ll \Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, 1 < q \leq 2.$$

Let us obtain the lower estimate. By virtue of proposition 9 and (1) we can write

$$e_M\left(L^{\psi}_{\beta,1}\right)_q = \sup_{f\in L^{\psi}_{\beta,1}} \inf_{\theta_M} \sup_{\substack{P^{\psi}_{\beta}\in L^{\perp}(\theta_M),\\ \|P^{\psi}_{\beta}\|_{q'}\leq 1}} \left|\int_{\pi_d} f(x)\overline{P^{\psi}_{\beta}(x)}dx\right| =$$
$$= \sup_{\|\varphi\|_1\leq 1} \inf_{\theta_M} \sup_{\substack{P^{\psi}_{\beta}\in L^{\perp}(\theta_M),\\ \|P^{\psi}_{\beta}\|_{q'}\leq 1}} \left|\int_{\pi_d}\left((2\pi)^{-d}\int_{\pi_d}\varphi(t)D^{\psi}_{\beta}(x-t)dt\right)\overline{P^{\psi}_{\beta}(x)}dx\right|. \tag{12}$$

Now we are going to verify the conditions of the Fubini theorem (see, e.g., [30, c. 336]) for an integral on the right side of (12). Let us consider the integral

$$\int_{\pi_d}\varphi(t)\left(\int_{\pi_d}D^{\psi}_{\beta}(x-t)\overline{P^{\psi}_{\beta}(x)}dx\right)dt. \tag{13}$$

Since $D_\beta^\psi \in L_q$, $1 < q < \infty$, and $P_\beta^\psi \in L_{q'}$, then using the Holder's inequality we get

$$\int\limits_{\pi_d} D_\beta^\psi(x-t)\overline{P_\beta^\psi(x)}dx \le \left\|D_\beta^\psi\right\|_q \left\|P_\beta^\psi\right\|_{q'},$$

and then for an arbitrary function $\varphi \in L_1$ the integral (13) is convergent.

After changing the order of integration in (12) we receive

$$e_M\left(L_{\beta,1}^\psi\right)_q = \sup_{\|\varphi\|_1\le 1} \inf_{\theta_M} \sup_{\substack{P_\beta^\psi \in L^\perp(\theta_M),\\ \|P_\beta^\psi\|_{q'}\le 1}} \int\limits_{\pi_d} \varphi(t)\times$$

$$\times\left((2\pi)^{-d}\int\limits_{\pi_d} D_\beta^\psi(x-t)\overline{P_\beta^\psi(x)}dx\right)dt.$$

Using first the Holder's inequality (if $p=1$, $p'=\infty$) and then proposition 9 we get

$$e_M\left(L_{\beta,1}^\psi\right)_q = \inf_{\theta_M} \sup_{\substack{P_\beta^\psi \in L^\perp(\theta_M),\\ \|P_\beta^\psi\|_{q'}\le 1}} \left\|(2\pi)^{-d}\int\limits_{\pi_d} D_\beta^\psi(x-t)\overline{P_\beta^\psi(x)}dx\right\|_\infty \ge$$

$$\ge \inf_{\theta_M} \sup_{\substack{P_\beta^\psi \in L^\perp(\theta_M),\\ \|P_\beta^\psi\|_{q'}\le 1}} \left|(2\pi)^{-d}\int\limits_{\pi_d} D_\beta^\psi(x-t)\overline{P_\beta^\psi(x)}dx\right| =$$

$$= (2\pi)^{-d} e_M\left(D_\beta^\psi\right)_q.$$

By virtue of theorem 7 we can write

$$e_M\left(L_{\beta,1}^\psi\right)_q \gg \Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \quad 1 < q \le 2.$$

The lower estimate and consequently theorem 2 is proven.

Remark 3. *The corresponding statement if* $\psi_j(|k_j|) = |k_j|^{-r_j}$, $r_j > 1 - \frac{1}{q}$, $1 < q \le 2$, $k_j \in \mathbb{Z}\backslash\{0\}$, $j = \overline{1,d}$, *was formulated by A.S. Romanyuk [17].*

4. Conclusions

The paper continues investigation of the approximative characteristics that where considered earlier by Temlyakov V. N., Stepanets A. I., Romanyuk A. S. and other mathematicians. Many results for the best M-term and orthogonal trigonometric approximations of classes of functions $B_{p,\theta}^r$, $W_{\beta,p}^r$, H_p^r are already obtained. Note that the great attention was paid to classes of functions of one variable. Nevertheless the problem of estimation of the best M-term approximations of classes $L_{\beta,1}^\psi$ of multivariate (ψ,β)-differentiable functions remained unsolved until now. We have obtained order relations of the quantities $e_M(f)_q$

for the concrete functions D^{ψ}_{β}, that are of interest themselves. And besides, by using established results, we have written down the order relations for classes $L^{\psi}_{\beta,1}$.

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ON OPTIMAL SELECTION OF GALERKIN'S INFORMATION FOR SOLVING SEVERELY ILL-POSED PROBLEMS

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Резюме. Для розв'язування експоненційно некоректних задач розроблено економічний проекційний метод, який полягає у комбінуванні стандартного метода Тіхонова та принципу нев'язки Морозова. При цьому встановлено, що запропонований алгоритм забезпечує оптимальний порядок інформаційної складності на класі досліджуваних задач.

Abstract. An economical projection method is developed for solving exponentially ill-posed problems. The method consist in combination of the standard Tikhonov method and the Morozov discrepancy principle. Herewith, it is established that this approach provides optimal order of information complexity on the class of problems under consideration.

1. Introduction

The implicit (a posteriori) choice of the regularization parameter without any information on smoothness of a desired solution is usually assume to be the key issue in the theory of ill-posed problems. It is well-known, there are a lot of different rules of a regularization parameter choice among them we mention discrepancy principle [6, 8, 9, 20], Gfrerer's method [3, 19], the monotone error rule [27], the balancing principle [2, 4, 14, 25] which sometimes is called the Lepskij principle. Nowadays, it is sure the discrepancy principle is the most common one.

In the present paper that is extension of the research started in [23, 24] the authors develop economical projection method for effective solving severely ill-posed problems. As a regularization the standard Tikhonov method is applied. Unlike to above-mentioned works, the regularization parameter is chosen a posteriori, namely, according with the balancing principle. Moreover, it is established that a proposed strategy maintains optimal oder accuracy on the class of problems under consideration, as well as provides oder estimates of the information complexity.

The organization of the material is as follows: in Section 2 we give the statement of the problem. Further in Section 3 the regularization and discretization methods are described. Auxiliary statements and facts are in Section 4. An algorithm of the regularization parameter choice by discrepancy principle is

Key words. Severely ill-posed problems; minimal radius of Galerkin information; discrapency principle; information complexity.

presented in Section 5. The combination of proposed methods allows to established optimal order accuracy for solving equations from the class of problems under research. Finally, in Section 6, the authors establish the main result. Namely, the order estimate for the minimal radius of the Galerkin information is obtained.

2. Statement of the problem

Following [23] we present the rough statement of the problem. Consider Fredholm's integral equation of the first kind

$$Ax(t) = f(t), \qquad t \in [0,1], \tag{1}$$

with

$$Ax(t) = \int_0^1 a(t,\tau)x(\tau)d\tau, \tag{2}$$

acting continuously in $L_2 = L_2(0,1)$. Suppose that Range(A) is not closed in L_2 and $f \in$ Range(A).

We also assume that a perturbation $f_\delta \in L_2 : \|f - f_\delta\| \le \delta, \quad \delta > 0$ is given instead of the right-hand side of the equation (1).

The problem (1) is regarded as severely ill-posed problem if its solution has substantially worse smoothness than a kernel $a(\cdot,\tau)$ In such case it is nature to assume that an exact solutions satisfies some logarithmic source condition, in other words it belongs to the set

$$M_p(A) := \{u : u = \ln^{-p}(A^*A)^{-1}v, \quad \|v\| \le \rho\},$$

where p, ρ are some positive parameters and A^* is adjoined operator to A. Such problems are called exponentially ill-posed (see e.g. [5]).

Note, that the exact information about smoothness, namely, the parameter p, is usually not available by practical experiment. For this reason the set

$$M(A) := \bigcup_{p\in(0,p_1]} M_p(A) \tag{3}$$

is considered in place of $M_p(A)$. Here $p_1 < \infty$ is an upper bound for possible values of p.

Within the framework of our researches we construct an approximation to the exact solution $x^\dagger$ (1), which has minimal norm in L_2 and belongs to the set $M(A)$. From now on, we assume that a parameter p is unknown.

Let $\{e_i\}_{i=1}^\infty$ be some orthonormal basis in L_2, and let P_m denotes the orthogonal projection onto span$\{e_1, e_2, \ldots, e_m\}$

$$P_m\varphi(t) = \sum_{i=1}^m (\varphi, e_i)e_i(t).$$

Consider the following class of operators (2):

$$\mathcal{H}_\gamma^{r,s} = \{A : \|A\| \le \gamma_0, \quad \sum_{n+m=1}^\infty \hat{a}_{n,m}^2\, \underline{n}^{2r}\underline{m}^{2s} \le \gamma_1^2\}, \qquad r,s > 0, \tag{4}$$

where

$$\hat{a}_{n,m} = \int_0^1 \int_0^1 e_n(t)a(t,\tau)e_m(\tau)d\tau dt,$$

$\gamma_0 \leq e^{-\frac{1}{2}}, \gamma = (\gamma_0; \gamma_1)$, $\underline{n} = 1$ if $n = 0$ and $\underline{n} = n$ otherwise.

If the kernel $a(t,\tau)$ of A has mixed partial derivatives and the inequalities

$$\int_0^1 \int_0^1 \left[\frac{\partial^{i+j} a(t;\tau)}{\partial t^i \partial \tau^j}\right]^2 dt d\tau < \infty$$

hold for all $i = 0, 1, \ldots, r$, $j = 0, 1, \ldots, s$ then it is known (see e.g. [16]), $A \in \mathcal{H}_\gamma^{r,s}$ for some $\gamma = (\gamma_0, \gamma_1)$.

From now on, class of equations (1) with operators belonging to $\mathcal{H}_\gamma^{r,s}$ (4) and solutions from $M(A)$ (3) will be denoted by $(\mathcal{H}_\gamma^{r,s}, M(A))$. In the present paper we concentrate on the study of projection methods for solving equations belonging to $(\mathcal{H}_\gamma^{r,s}, M(A)), \quad r \geq s$.

A discretization projection scheme of equations (1) with the perturbed right-hand side one can define by means of a finite set of the inner products

$$(Ae_j, e_i), \qquad (i,j) \in \Omega, \tag{5}$$

$$(f_\delta, e_k), \qquad k \in \omega_1, \quad \omega_1 = \{i\colon (i,j) \in \Omega\}, \tag{6}$$

where Ω to be an bounded domain of the coordinate plane $[1,\infty) \times [1,\infty)$. The inner products (5), (6) are used to call the Galerkin information about (1). Here $\mathrm{card}(\Omega)$ is the total number of the inner products (5). In particular, if $\Omega = [1,n] \times [1,m]$, then one deal with the standard Galerkin discretization scheme, $\mathrm{card}(\Omega) = n \cdot m$. Researches for various classes of ill-posed problems related to such scheme of discretization were conducted in a number of works among which we mention [7, 17, 18].

Definition 11. A projection method of solving (1) can be associated with any mapping $\mathcal{P} = \mathcal{P}(\Omega) : L_2 \to L_2$ which by the Galerkin information (5), (6) about (1) provides a correspondence between the right-hand side of the equation being solved and an element $\mathcal{P}(A_\Omega)f_\delta \in L_2$, which is a polynomial by the basis $\{e_i\}_{i=1}^\infty$ with harmonic numbers from $\omega_2 := \{j\colon (i,j) \in \Omega\}$. This element is taken as an approximate solution (1).

The error of the method $\mathcal{P}(\Omega)$ on the class of equations $(\mathcal{H}_\gamma^{r,s}, M_p(A))$ is defined as

$$e_\delta\left(\mathcal{H}_\gamma^{r,s}, M(A), \mathcal{P}(\Omega)\right) = \sup_{A \in \mathcal{H}_\gamma^{r,s}} \sup_{x^\dagger \in M(A)} \sup_{f_\delta : \|f - f_\delta\| \leq \delta} \|x^\dagger - \mathcal{P}(A_\Omega)f_\delta\|.$$

The minimal radius of the Galerkin information is given by

$$R_{N,\delta}\left(\mathcal{H}_\gamma^{r,s}, M(A)\right) = \inf_{\Omega\colon \mathrm{card}(\Omega) \leq N} \inf_{\mathcal{P}(\Omega)} e_\delta\left(\mathcal{H}_\gamma^{r,s}, M(A), \mathcal{P}(\Omega)\right).$$

This value describes the minimal possible accuracy (among all projection methods), while the Galerkin information amount are bound. Thus, $R_{N,\delta}$ characterizes information complexity on the class of problems $(\mathcal{H}_\gamma^{r,s}, M(A))$.

It is easy to see, that such studies belong to the range of problems from Information Based Complexity Theory. The fundamentals of this theory were introduced in monographs [28, 29]. It should be noted that in recent years the interest to such researches in the light of ill-posed problems is greatly increase. In the work [18] first economical projection methods for solving moderately ill-posed problems were constructed. The standard Galerkin scheme was employed as discretization scheme. But first order estimates for complexity of moderately ill-posed problems were obtained in [16, 21, 22]. The authors point to the fact that optimal orders of such values are achieved under a modified Galerkin scheme that is called hyperbolic cross. The complexity of severely ill-posed problems began to be study relatively recently. These researches are highlighted in the series of works, we mention [7, 23, 24].

In the present paper as opposite to above-mentioned one, an economical projection scheme with a posteriori rule of regularization parameter choice will be developed for solving severely ill-posed problems.

3. Regularization and discretization strategies

To guarantee stable approximations we apply the standard Tikhonov method. By means of this method the rugularized solution x_α is defined as the solution of the variation problem

$$I_\alpha(x) := \|Ax - f_\delta\|^2 + \alpha\|x\|^2 \to \min . \qquad (7)$$

For a numerical realization of the standard Tikhonov method it is necessary to carry out all computations with finite amount of input data. For that reason the variation problem (7) is replaced by following

$$I_{\alpha,n}(x) = \|A_n x - f_\delta\|^2 + \alpha\|x\|^2 \to \min,$$

where A_n is some operator of the finite rank.

The idea to apply the hyperbolic cross to operator equations of the second kind belongs to S.V. Pereverzev and implements in the series of works (see e.g. [10–13]). The efficiency of the hyperbolic cross for ill-posed problems has been demonstrated in [15, 16, 23]. Within the framework of our researches we apply a projection scheme with $\Omega = \Gamma_n^a$, where

$$\Gamma_n^a = \{1\} \times [1; 2^{2an}] \bigcup_{k=1}^{2n} (2^{k-1}; 2^k] \times [1; 2^{(2n-k)a}] \subset [1; 2^{2n}] \times [1; 2^{2an}] \qquad (8)$$

is a hyperbolic cross on the coordinate plane by the basis $\{e_i\}_{i=1}^\infty$ involved in the definition of the class $\mathcal{H}_\gamma^{r,s}$. Here for $r > s$ the parameter a is an arbitrary real number such that $1 < a < \frac{r}{s}$, and for $a = 1$ we set $r = s$. To simplify computations we assume that ak are integer numbers. An approximate solution one can find from an operator equation of the second kind

$$\alpha x + A_n^* A_n x = A_n^* f_\delta .$$

On other words, we seek an approximate solution $x = x_{\alpha,n}^\delta$ of the form

$$x_{\alpha,n}^\delta = g_\alpha(A_n^* A_n) A_n^* f_\delta , \qquad (9)$$

where $g_\alpha(\lambda) = (\alpha + \lambda)^{-1}$, and

$$A_n = P_1 A P_{2^{2an}} + \sum_{k=1}^{2n} \left(P_{2^k} - P_{2^{k-1}}\right) A P_{2^{(2n-k)a}}. \qquad (10)$$

Moreover we introduce following auxiliary elements

$$x_\alpha = g_\alpha(A^* A) A^* f, \qquad (11)$$

$$x_{\alpha,n} = g_\alpha(A_n^* A_n) A_n^* f. \qquad (12)$$

4. Auxiliary results

In this Section we formulate some definitions and facts, and also the series of auxiliary assertions which shell later need.

It is well-known (see e.g. [30]), that for any linear bounded operator A the inequalities

$$\begin{gathered} \|(\alpha I + A^* A)^{-1}\| \le \alpha^{-1}, \quad \|(\alpha I + A^* A)^{-1} A^*\| \le \tfrac{1}{2\sqrt{\alpha}}, \\ \|A(\alpha I + A^* A)^{-1} A^*\| \le 1 \end{gathered} \qquad (13)$$

hold.

Lemma 1. *(see [30, p. 34]) If g to be bounded Borel measurable function on $[0; \gamma_0^2]$, $A \in \mathcal{L}(L_2, L_2)$, $\|A\| \le \gamma_0$, then*

$$\begin{gathered} A^* g(AA^*) = g(A^* A) A^*, \\ A g(A^* A) = g(AA^*) A. \end{gathered} \qquad (14)$$

Lemma 2. *(see [20]) Let $\|A\| \le \gamma_0 \le e^{-1/2}$. Then for sufficiently small $\alpha \in (0, e^{-2p})$ it holds*

$$\|Ax_\alpha - f\| \le \gamma_0^{-1} \rho \sqrt{\alpha} \ln^{-p} 1/\alpha,$$

where x_α is determined by (11).

Lemma 3. *(see [20]) Let $\|A\| \le \gamma_0 \le e^{-1/2}$, and α is such that*

$$\|Ax_\alpha - f\| \le d' \delta,$$

where $d' > 0$ is some positive constant. Then the estimate

$$\|x^\dagger - x_\alpha\| \le \xi \ln^{-p} 1/\delta$$

is fulfilled. The constant $\xi > 0$ depends only on d', ρ and p.

Lemma 4. *For any $\alpha > 0$ and $n \in \mathbb{N}$ the estimate*

$$\|Ax_\alpha - f\| \le \|A_n x_{\alpha,n}^\delta - P_{2^{2n}} f_\delta\| + \left(\|(I - P_{2^{2n}}) f\|^2 + \delta^2\right)^{1/2} + \frac{5}{4} \rho \|A - A_n\|$$

holds, where x_α and $x_{\alpha,n}^\delta$ is determined by (11) and (9), respectively.

Proof. First off all, we note that

$$\|x^\dagger\| = \|\ln^{-p}(A^*A)v\| \leq \rho \sup_{0<\lambda\leq\gamma_0^2} |\ln^{-p} 1/\lambda| \leq \rho. \tag{15}$$

Further, consider the decomposition

$$Ax_\alpha - f = A_n x_{\alpha,n}^\delta - P_{2^{2n}} f_\delta + S_1 + S_2,$$

where

$$S_1 := -\left(I - A_n g_\alpha(A_n^* A_n) A_n^*\right)\left(f - P_{2^{2n}} f_\delta\right),$$
$$S_2 := \left(A g_\alpha(A^*A)A^* - A_n g_\alpha(A_n^* A_n) A_n^*\right) f.$$

Now we are going to bound each term S_1, S_2. By (13), (14) we immediate find

$$\|S_1\| \leq \|I - A_n(\alpha I + A_n^* A_n)^{-1} A_n^*\| \|f - P_{2^{2n}} f_\delta\| \leq$$
$$\leq \|I - (\alpha I + A_n A_n^*)^{-1} A_n A_n^*\| \|(I - P_{2^{2n}}) f + P_{2^{2n}}(f - f_\delta)\| \leq$$
$$\leq \left(\|\left(I - P_{2^{2n}}\right) f\|^2 + \delta^2\right)^{\frac{1}{2}}.$$

It remains to estimate the norm of S_2. First, rewrite S_2 as follows

$$S_2 = \left(A g_\alpha(A^*A)A^* - A_n g_\alpha(A_n^* A_n) A_n^*\right) f =$$
$$= \alpha\left(\alpha I + A_n A_n^*\right)^{-1}\left(AA^* - A_n A_n^*\right)\left(\alpha I + AA^*\right)^{-1} f = \overline{s}_1 + \overline{s}_2,$$

where

$$\overline{s}_1 := \alpha\left(\alpha I + A_n A_n^*\right)^{-1}\left(A - A_n\right) A^* \left(\alpha I + AA^*\right)^{-1} A x^\dagger,$$
$$\overline{s}_2 := \alpha\left(\alpha I + A_n A_n^*\right)^{-1} A_n \left(A^* - A_n^*\right)\left(\alpha I + AA^*\right)^{-1} A x^\dagger.$$

Further, we bound norms of $\overline{s}_1$ and $\overline{s}_2$. By (13), (14) and (15) we obtain

$$\|\overline{s}_1\| \leq \alpha\|\left(\alpha I + A_n A_n^*\right)^{-1}\| \|A - A_n\| \|\left(\alpha I + A^*A\right)^{-1} A^*A\| \|x^\dagger\| \leq$$
$$\leq \rho\|A - A_n\|,$$
$$\|\overline{s}_2\| \leq \alpha\|\left(\alpha I + A_n A_n^*\right)^{-1} A_n\| \|A^* - A_n^*\| \|\left(\alpha I + AA^*\right)^{-1} A\| \|x^\dagger\| \leq$$
$$\leq \frac{\rho}{4}\|A - A_n\|.$$

Thus,

$$\|S_2\| \leq \|\overline{s}_1\| + \|\overline{s}_2\| \leq \frac{5\rho}{4}\|A - A_n\|.$$

Summing up the above bounds, we finally get

$$\|Ax_\alpha - f\| \leq \|A_n x_{\alpha,n}^\delta - P_{2^{2n}} f_\delta\| +$$
$$+ \left(\|(I - P_{2^{2n}}) f\|^2 + \delta^2\right)^{1/2} + \frac{5\rho}{4}\|A - A_n\|.$$

The lemma is proved. □

Lemma 5. *The two-side estimates*

$$2^{2n} n < \operatorname{card}(\Gamma_n^1) \leq 2 \cdot 2^{2n} n, \qquad r = s,$$
$$\eta_1 2^{2an} \leq \operatorname{card}(\Gamma_n^a) \leq \eta_2 2^{2an}, \qquad r > s, \tag{16}$$

are hold, with $\eta_1 = 1 + \frac{1 - 2^{3(1-a)}}{1 - 2^{1-a}}$, $\eta_2 = \frac{2 - 2^{1-a}}{1 - 2^{1-a}}$.

Proof. From (8) it follows

$$\mathrm{card}(\Gamma_n^a) = \sum_{k=0}^{2n} \mathrm{card}(Q_k),$$

where

$$Q_k = \begin{cases} (2^{k-1}; 2^k] \times [1; 2^{(2n-k)a}], & k = 1, 2, \dots, 2n \\ \{1\} \times [1; 2^{2an}], & k = 0 \end{cases},$$

and we obtain

$$\mathrm{card}(\Gamma_n^a) = 2^{2an} + \frac{1}{2}\sum_{k=1}^{2n} 2^k 2^{(2n-k)a}.$$

Further, consider two cases. It is obvious that for $r = s$ it holds

$$\mathrm{card}(\Gamma_n^1) = 2^{2n} + \frac{1}{2}\sum_{k=1}^{2n} 2^{2n} = 2^{2n}\left(1+n\right) = 2^{2n} n \left(1 + \frac{1}{n}\right).$$

Hence,

$$2^{2n} n < \mathrm{card}(\Gamma_n^1) \le 2 \cdot 2^{2n} n.$$

When $r > s$ the sequence $\{\mathrm{card}(Q_k)\}_{k=1}^{2n}$ is the geometric progression with the quotient 2^{1-a}, and the relation

$$\mathrm{card}(\Gamma_n^a) = 2^{2an}\left(1 + \frac{1}{2}\sum_{k=1}^{2n} 2^{k(1-a)}\right)$$

is hold. It follows that

$$\mathrm{card}(\Gamma_n^a) = \frac{1}{2} 2^{2an}\left(1 + \sum_{k=0}^{2n} 2^{k(1-a)}\right) = \frac{1}{2} 2^{2an}\left(1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}}\right).$$

Further, we obtain lower and upper bounds for the bracketed expression:

$$1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}} = \frac{2 - 2^{1-a}\left(1 + 2^{(1-a)2n}\right)}{1 - 2^{1-a}} \le \frac{2 - 2^{1-a}}{1 - 2^{1-a}},$$

$$1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}} \ge 1 + \frac{1 - 2^{3(1-a)}}{1 - 2^{1-a}}.$$

Thus, finally we get

$$\left(1 + \frac{1 - 2^{3(1-a)}}{1 - 2^{1-a}}\right) 2^{2an} \le \mathrm{card}(\Gamma_n^a) \le \frac{2 - 2^{1-a}}{1 - 2^{1-a}} 2^{2an}.$$

The statement of the lemma is proved. □

It is known (see. [21]), that for any $A \in \mathcal{H}_\gamma^{r,s}$ the inequality

$$\|A - A_n\| \le \varepsilon_{r,s}(n) \tag{17}$$

is fulfilled, where

$$\varepsilon_{r,s}(n) = \begin{cases} \gamma_1 2^{r+1/2}\sqrt{n}2^{-2rn}, & r = s \\ \gamma_1 \left(1 + \frac{2^r}{1-2^{as-r}}\right) 2^{-2nas}, & r > s \end{cases}.$$

5. Error estimate of the algorithm

5.1. **Algorithm (Discrepancy principle as stop rule).** Let us fix $\theta \in (0,1)$ and $\alpha_0 \in (0,1]$. We are going to choose the regularization parameter α according with the rule

$$\alpha \in \Delta_\theta(\delta) = \{ \quad \alpha\colon \alpha = \alpha_m := \alpha_0\theta^m, \quad m = 0,1,2,\dots, \quad \alpha \in (\delta^2, \alpha_0]\}, \tag{18}$$

and the discretization parameter n as follows

$$\varepsilon_{r,s}(n) = \frac{4}{5\rho}\delta. \tag{19}$$

Now, we describe proposed algorithm with the discrepancy principle as a stop rule concerning to studied problem.

1. Input data: $A \in \mathcal{H}_\gamma^{r,s}$, f_δ, δ, ρ.
2. To construct A_n (10) and $P_{2^{2n}} f_\delta$ we compute the inner products (5), (6).
3. The cycle: $\quad m = 1,2,\dots,M, \qquad \alpha = \alpha_m = \alpha_0\theta^m$.
   An approximate solution $x^\delta_{\alpha_m,n}$ (9) is computed by solving the equation
   $$\alpha_m x^\delta_{\alpha_m,n} + A_n^* A_n x^\delta_{\alpha_m,n} = A_n A^* f_\delta.$$
   The cycle is running as long as stop rule conditions will be meet.
4. The stop rule (the discrepancy principle)
   $$\|A_n x^\delta_{\alpha_M,n} - P_{2^{2n}} f_\delta\| \le d\delta, \tag{20}$$
   $$\|A_n x^\delta_{\alpha_m,n} - P_{2^{2n}} f_\delta\| > d\delta, \tag{21}$$
   with $m < M$, $\quad d > \sqrt{2}+1$, and $x^\delta_{\alpha_M,n}$ is determined by (9).

Introduced projection method (10), (18)–(21) we denoted as $\mathcal{P}$‘.

Lemma 6. *Let α_M such that the conditions (20) and (21) are satisfied with $d > \sqrt{2}+1$, and the parameter n in (10) is chosen as (19). Then there are the constants $d_1, d_2 > 0$, that the two-side estimate*

$$d_1\delta \le \|Ax_{\alpha_M} - f\| \le d_2\delta$$

is fulfilled.

Proof. First, note that by (17) and (19) it holds

$$\begin{aligned} \frac{5\rho}{4}\|A - A_n\| &\le \delta, \\ \|(I - P_{2^{2n}})f\| &\le \delta. \end{aligned} \tag{22}$$

If α_M meets the condition (20) then

$$\|A_n g_{\alpha_M}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| \le d\delta,$$

and applying Lemma 4 we obtain

$$\|Ax_{\alpha_M} - f\| \le d\delta + \sqrt{2\delta^2} + \delta = (d + \sqrt{2} + 1)\delta.$$

At the same time, kipping in mind (21), for $\alpha = \alpha_{M-1}$ we have

$$\|A_n g_{\alpha_{M-1}}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| > d\delta. \tag{23}$$

Owing to the inverse triangle rule it holds

$$\|Ax_{\alpha_{M-1}} - f\| \ge \|A_n g_{\alpha_{M-1}}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| - (\sqrt{2} + 1)\delta. \tag{24}$$

By spectral decomposition of the operator A we get

$$\begin{aligned}\|Ax_{\alpha_M} - f\|^2 &= \sum_{k=1}^{\infty} \lambda_k^2 \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2 \left[\frac{\lambda_k^2}{\alpha_M + \lambda_k^2} - 1\right]^2 = \\ &= \alpha_M^2 \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\left(\alpha_M + \lambda_k^2\right)^2} \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2 > \\ &> \theta^2 \alpha_{M-1}^2 \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\left(\alpha_{M-1} + \lambda_k^2\right)^2} \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2.\end{aligned}$$

Hence,

$$\|Ax_{\alpha_M} - f\|^2 > \theta^2 \|Ax_{\alpha_{M-1}} - f\|^2. \tag{25}$$

Substituting (23) and (24) in (25), we finally obtain

$$\|Ax_{\alpha_M} - f\| \ge \theta(d - \sqrt{2} - 1)\delta.$$

Thus, the lemma is proved with $d_1 = \theta(d - \sqrt{2} - 1)\delta$ and $d_2 = \theta(d + \sqrt{2} + 1)\delta$.

5.2. **Error estimate of the algorithm $\mathcal{P}'$.**

Theorem 1. *Let $\|A\| \le \gamma_0 \le e^{-1/2}$, the parameters of regularization α_M and discretization n are chosen as in (20) and (19), correspondingly. Than the estimate*

$$\|x^\dagger - x_{\alpha_M,n}^\delta\| \le \tilde{c} \ln^{-p} 1/\delta \tag{26}$$

holds, where the constant $\tilde{c} > 0$ only depends on γ_0, d_1, d_2, ρ and p; $x_{\alpha_M,n}^\delta$ is determined by (9).

Proof. It is obvious that

$$\|x^\dagger - x_{\alpha_M,n}^\delta\| \le \|x^\dagger - x_{\alpha_M}\| + \|x_{\alpha_M} - x_{\alpha_M,n}\| + \|x_{\alpha_M,n} - x_{\alpha_M,n}^\delta\|.$$

Owing to 3 for the first term we have

$$\|x^\dagger - x_{\alpha_M}\| \le \xi \ln^{-p} 1/\delta.$$

By applying (13) the last term is immediately bounded

$$\|x_{\alpha_M,n} - x_{\alpha_M,n}^\delta\| = \|(\alpha_M I + A_n^* A_n)^{-1} A_n^* (f - f_\delta)\| \le \frac{\delta}{2\sqrt{\alpha_M}}.$$

Finally, we need to estimate the second term. First, consider the decomposition

$$\begin{aligned}x_{\alpha_M} - x_{\alpha_M,n} &= (\alpha_M I + A^* A)^{-1} A^* A x^\dagger - (\alpha_M I + A_n^* A_n)^{-1} A_n^* A x^\dagger = \\ &= T_1 x^\dagger + T_2 x^\dagger,\end{aligned} \tag{27}$$

where

$$T_1 := (\alpha_M I + A^*A)^{-1}A^*A - (\alpha_M I + A_n^*A_n)^{-1}A_n^*A_n,$$
$$T_2 := (\alpha_M I + A_n^*A_n)^{-1}A_n^*(A_n - A).$$

By (13), (19) and (17) we have

$$\|T_2\| \le \frac{1}{2\sqrt{\alpha_M}}\frac{4}{5\rho}\delta = \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

It is remain to estimate $\|T_1\|$. Due to (14), we rewrite T_1 as follows

$$T_1 = \alpha_M\,(\alpha_M I + A^*A)^{-1}\,(A^*A - A_n^*A_n)\,(\alpha_M I + A_n^*A_n)^{-1} = \overline{T}_1 + \overline{T}_2,$$

where

$$\overline{T}_1 := \alpha_M\,(\alpha_M I + A^*A)^{-1}\,A^*\,(A - A_n)\,(\alpha_M I + A_n^*A_n)^{-1}\,,$$
$$\overline{T}_2 := \alpha_M\,(\alpha_M I + A^*A)^{-1}\,(A^* - A_n^*)\,A_n\,(\alpha_M I + A_n^*A_n)^{-1}\,.$$

Further, we estimate the norms of $\overline{T}_1$ and $\overline{T}_2$. Owing to (13), (19) and (17) the norm of $\overline{T}_1$ is immediately bounded as

$$\|\overline{T}_1\| \le \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Now, we are going to estimate the norm of $\overline{T}_2$. By (14) we have

$$\overline{T}_2 = \alpha_M\,(\alpha_M I + A^*A)^{-1}\,(A^* - A_n^*)\,(\alpha_M I + A_nA_n^*)^{-1}\,A_n.$$

Applying (13), (19) and (17), we obtain

$$\|\overline{T}_2\| \le \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Hence,

$$\|T_1\| \le \|\overline{T}_1\| + \|\overline{T}_2\| \le \frac{4}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Thus,

$$\|x_{\alpha_M} - x_{\alpha_M,n}\| \le \frac{6}{5}\frac{\delta}{\sqrt{\alpha_M}}.$$

Summing up the above bounds we finally get

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le \xi \ln^{-p} 1/\delta + \frac{6}{5}\frac{\delta}{\sqrt{\alpha_M}} + \frac{1}{2}\frac{\delta}{\sqrt{\alpha_M}} \le \xi \ln^{-p} 1/\delta + \frac{17}{10}\frac{\delta}{\sqrt{\alpha_M}}.$$

Further, if α_M is chosen as in (20) and the inequality $\alpha_M \ge \delta$ holds then for sufficiently small δ we have

$$\|x_\dagger - x^\delta_{\alpha_M,n}\| \le \xi \ln^{-p} 1/\delta + \frac{17}{10}\sqrt{\delta} \le \tilde{c}_1 \ln^{-p} 1/\delta,$$

with $\tilde{c}_1 = \xi + \frac{17}{10}$.

Otherwise, if $\alpha_M \le \delta$ then by Lemma 2 and Lemma 6 we get

$$d_1\delta \le \|Ax_{\alpha_M} - f\| \le \gamma_0^{-1}\rho\sqrt{\alpha_M}\ln^{-p} 1/\alpha_M \le \gamma_0^{-1}\rho\sqrt{\alpha_M}\ln^{-p} 1/\delta.$$

Thus,

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le \xi \ln^{-p} 1/\delta + \frac{17}{10}\frac{\gamma_0^{-1}\rho}{d_1} \ln^{-p} 1/\delta = \tilde{c}_2 \ln^{-p} 1/\delta,$$

where $\tilde{c}_2 = \xi + \frac{17}{10}\frac{\gamma_0^{-1}\rho}{d_1}$. The theorem is proved with $\tilde{c} = \max\{\tilde{c}_1, \tilde{c}_2\}$. □

6. Minimal radius of Galerkin's information. Optimal order estimate

Theorem 2. *For sufficiently small δ the estimate*

$$R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le e_\delta\left(\mathcal{H}^{r,s}_\gamma M(A), \mathcal{P}^{‘}\right) \le c_p \ln^{-p} N^{2s}$$

is fulfilled, where $c_p > 0$ depends only on $\gamma, r, s, d_1, d_2, \rho$ and p. Moreover,

$$\mathrm{card}(\Gamma^a_n) \asymp \begin{cases} \delta^{-\frac{1}{r}}(\ln \delta^{-1})^{1+\frac{1}{2r}}, & r = s,\\ delta^{-\frac{1}{s}}, & r > s. \end{cases}$$

Proof. Rewrite the right-hand side of (26) by N, where

$$N = \begin{cases} c_1' n 2^{2n}, & r = s,\\ c_2' 2^{2an}, & r > s, \end{cases}$$

$1 < c_1' \le 2$, $\quad 1 + \frac{1-2^{3(1-a)}}{1-2^{1-a}} \le c_2' \le \frac{2-2^{1-a}}{1-2^{1-a}}$ (see Lemma 5). Further, we consider two cases.

First, let $r = s$. Owing to (16),(19) we have

$$\delta^{-1} = \frac{4}{5\rho\bar{c}_1} n^{-1/2} 2^{2rn} = \frac{4(c_1')^{-r}}{5\rho\bar{c}_1} N^r n^{-\frac{1}{2}-r}, \tag{28}$$

with $\bar{c}_1 = \gamma_1 2^{r+1/2}$. It is easy to see that $\ln N = \ln c_1' + 2n\ln 2 + \ln n$. It follows $n \le \frac{\ln N}{2\ln 2}$. Kipping in the mind the last inequality, from (28) we obtain the lower bound of δ^{-1}

$$\delta^{-1} \ge \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_1} N^r (\ln N)^{-1/2-r}.$$

For any $\mu > 0$ there are some N_0 that for all $N \ge N_0$ it holds $\ln N \le N^\mu$. Hence,

$$\delta^{-1} \ge \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_1} N^r N^{\mu(-1/2-r)} = $$
$$= \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_2} N^{(1-\mu)r-\frac{1}{2}\mu}.$$

There are always exist μ such that $(1-\mu)r - \frac{1}{2}\mu > 0$, and the estimate (26) we can rewrite as follows

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_{p,1} \ln^{-p} N^{2r}. \tag{29}$$

Now, we are going to consider the case $r > s$. Using the same arguments as above, by (16) and (19) we have

$$\delta^{-1} = \frac{4}{5\rho\bar{c}_2} 2^{2asn} = \frac{4(c_2')^{-s}}{5\rho\bar{c}_2} N^s, \tag{30}$$

where

$$\bar{c}_2 = \gamma_1 \left(1 + \frac{2^r}{1 - 2^{as-r}}\right).$$

In this case the estimate (26) we rewrite as follows

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_{p,2} \ln^{-p} N^{2s}. \tag{31}$$

Taking into account the definition $R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right)$, and also the relations (29) and (31) we have

$$R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le \|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_p \ln^{-p} N^{2s},$$

where $c_p = \max\{c_{p,1}, c_{p,2}\}$.

It is remain to express the amount $\mathrm{card}(\Gamma^a_n)$ by δ. Let consider the two cases. First let $r = s$, then

$$\mathrm{card}(\Gamma^1_n) := N \asymp 2^{2n} n = (\sqrt{n} 2^{-2sn})^{-\frac{1}{s}} n^{1+\frac{1}{2s}} \asymp \delta^{-\frac{1}{s}} (\ln \delta^{-1})^{1+\frac{1}{2s}}.$$

2) Now let $r > s$, then

$$\mathrm{card}(\Gamma^a_n) := N \asymp 2^{2an} = (2^{-2asn})^{-\frac{1}{s}} \asymp \delta^{-\frac{1}{s}}.$$

Thus, summing up obtained estimates of $\mathrm{card}(\Gamma^a_n)$, we have

$$\mathrm{card}(\Gamma^a_n) \asymp \begin{cases} \delta^{-\frac{1}{r}} (\ln \delta^{-1})^{1+\frac{1}{2r}}, & r = s \\ \delta^{-\frac{1}{s}}, & r > s \end{cases}.$$

The statement of the theorem is completely proved. □

Below we formulate a result giving the order estimate of the minimal radius of the Galerkin information.

Theorem 3. *The two-side estimate*

$$\frac{1}{2^{p+1}} \ln^{-p} N^{2s} \le R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le c_p \ln^{-p} N^{2s}$$

holds. The indicate optimal order is achieved under the algorithm $\mathcal{P}$‘ (10), (18)–(21).

The lower bound for $R_{N,\delta}$ is established in [26], and the upper estimate was obtained in Theorem 2.

Remark 4. *Comparing results of Theorem 3 to that of [26], where the balancing principle was applied as stop rule, we can conclude that both approaches are achieved an optimal order of accuracy. Moreover, the proposed algorithm allows to provide order estimates on more wide classes of problems. Herewith, we reduce the amount of the Galerkin information (on the logarithmic multiplier) when $r = s$.*

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NUMERICAL SOLUTION OF LORD-SHULMAN THERMOPIEZOELECTRICITY FORCED VIBRATIONS PROBLEM

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Резюме. Ми розглядаємо модель термоп'єзоелектрики Лорда-Шульмана (LS). Для початково-крайової задачі LS-термоп'єзоелектрики формулюється відповідна варіаційна задача. Далі розглядаються вимушені коливання піроелектрика і варіаційна задача переписується у спеціальному вигляді для цього окремого випадку. Доводиться коректність останньої варіаційної задачі. З використанням дискретизації Гальоркіна будується чисельна схема для розв'язування цієї варіаційної задачі. Питання збіжності цієї схеми також розглянуті в цій статті. Зрештою, проводиться чисельний експеримент, який добре ілюструє вплив параметра "часу релаксації" на отримані розв'язки.

Abstract. We consider the Lord-Shulman (LS) model of thermopiezoelectricity. Variational formulation is constructed for the initial boundary value problem of LS-thermopiezoelectricity. Then forced vibrations of pyroelectric specimen are considered and the variational problem is rewritten in the special form for that particular case. Well-posedness of the latter variational problem is proved. Then using Galerkin semidiscretization a numerical scheme for solving this variational problem is built. The questions of convergence of this scheme are also covered in this article. Finally, a numerical experiment is performed, which perfectly illustrates the influence of "relaxation time" parameter on the obtained solutions.

1. Introduction

Nowadays piezoelectric and pyroelectric materials are widely utilized in various modern devices such as sensors, actuators, transducers, etc [14]. The classic theory of linear thermopiezoelectricity was introduced by Mindlin [12]. The further study of the theory was performed by Nowacki [13]. The main drawback of the classic theory is the assumption of infinite speed of propagation of thermal signals in the piezoelectric specimen. To overcome this, Lord and Shulman [10] proposed a modified theory of thermoelasticity (LS-theory), where the classic Fourier' law of heat conduction is replaced by Maxwell-Cattaneo equation with introduction of so-called "relaxation time". Chandrasekharaiah was the first researcher to apply the LS-theory to thermopiezoelectricity [5]. Later a set of generalization theories for thermoelasticity and thermopiezoelectricity

Key words. generalized thermopiezoelectricity; Lord-Shulman model; PZT-4 ceramics; thermoelectromechanical waves; harmonic forced vibrations; Galerkin method; finite element method.

was developed, for example Green-Lindsay, Chandrasekharaiah-Tzou, Green-Naghdi, etc. A good review of the existing generalization theories can be found in [1], [6], [8], [9]. Different methods were used by researchers to obtain the solutions of the generalized thermopiezoelectricity problem, see [2], [3], [7], [15], [20].

Forced vibrations of pyroelectrics is the special case of the thermopiezoelectricity problem and was studied under the classic (Mindlin's) theory in [11], [21] and [22]. In our previous work [19], we utilized our finite-element-based numerical scheme for solving forced vibrations problem under classic thermopiezoelectricity theory and developed an adaptive algorithm for obtaining solution with a preset level of accuracy. The goal of the present research is to construct a similar FEM-based numerical scheme for forced vibrations problem under LS-thermopiezoelectricity theory.

2. Problem statement

The theory of thermopiezoelectricity describes the coupled interaction of mechanical, electrical and thermal fields in pyroelectric material.

Suppose the piezoelectric specimen occupies a bounded domain Ω in Euclidean space $R^d, d = 1, 2,$ or 3 with continuous by Lipschitz boundary Γ with unit external normal vector $n = \{n_i\}_{i=1}^d$, where $n_i = cos(n, x_i)$. According to the classic theory (see [12, 13, 16, 17]), we need to find elastic displacement vector $\boldsymbol{u} = \boldsymbol{u}(x, t)$, electric potential $p = p(x, t)$ and temperature increment $\theta = \theta(x, t)$, which satisfy the following equations:

$$\rho u_i'' - \sigma_{ij,j} = \rho f_i, \tag{1}$$

$$D'_{k,k} + J_{k,k} = 0, \tag{2}$$

$$\rho(T_0 S' - w) + q_{i,i} = 0, \tag{3}$$

namely, equation of motion, differentiated Maxwell's equation and generalized heat equation respectively, where f_i is a vector of volume mechanical forces and w represents volume heat forces. Here the constitutive equations for stress tensor

$$\sigma_{ij} = c_{ijkm}[\varepsilon_{km} - \alpha_{km}\theta] - e_{kij}E_k, \tag{4}$$

electric displacement vector

$$D_k = e_{kij}\varepsilon_{ij} + \chi_{km}E_m + \pi_k\theta, \tag{5}$$

and entropy density

$$\rho S = c_{ijkm}\alpha_{km}\varepsilon_{ij} + \pi_k E_k + \tfrac{\rho c_v}{T_0}\theta \tag{6}$$

are used.

Vector J_k is the electrical current density, generated by a free electrical charge density. We assume that pyroelectric material is not an ideal dielectric, and the electric current runs through the pyroelectric specimen and satisfies standard Ohm's law, i.e.

$$J_k = z_{km}E_m(p). \tag{7}$$

Heat flux vector $\boldsymbol{q} = \boldsymbol{q}(x, t)$ is assumed to satisfy the standard Fourier's law:

$$q_i = -\lambda_{ij}\theta_{,j}. \tag{8}$$

Strain tensor ε_{km} and electrical field vector E_k are assumed to satisfy the relations

$$\begin{array}{l} \varepsilon_{km} = \varepsilon_{km}(\boldsymbol{u}) = \frac{1}{2}(u_{k,m} + u_{m,k}), \\ E_k = E_k(p) = -p_{,k}, \end{array} \tag{9}$$

where comma in the subscript stands for the partial derivative by the spatial variable, i. e. $g_{,k} = -\partial g / \partial x_k$.

The other symbols in the above equations represent the material properties of pyroelectric medium: c_{ijkm} is an elasticity coefficients tensor with common properties of symmetry and ellipticity, that is:

$$\begin{array}{l} c_{ijkm} = c_{jikm} = c_{kmij}, \\ c_{ijkm}\kappa_{ij}\kappa_{km} \geq c_0\kappa_{ij}\kappa_{km}, c_0 = const > 0, \quad \forall \kappa_{ij} = \kappa_{ji} \in R, \end{array} \tag{10}$$

α_{ij} is a thermal expansion tensor with similar properties

$$\begin{array}{l} \alpha_{ij} = \alpha_{ji}, \\ \alpha_{ij}\xi_i\xi_j \geq \alpha_0\xi_i\xi_j, \alpha_0 = const > 0, \quad \forall \xi_i \in R, \end{array} \tag{11}$$

e_{kij} ia a piezoelectricity tensor with properties:

$$e_{kij} = e_{kji}, \tag{12}$$

χ_{ij} is a dielectric permittivity tensor with properties

$$\begin{array}{l} \chi_{ij} = \chi_{ji}, \\ \chi_{ij}\xi_i\xi_j \geq \chi_0\xi_i\xi_i, \chi_0 = const > 0, \quad \forall \xi_i \in R, \end{array} \tag{13}$$

π_k are the pyroelectric coefficients, which are assumed to satisfy the following inequality, mentioned in [13]

$$\chi_{km}y_k y_m + 2\pi_k y_k \xi + \rho c_v \xi^2 \geq 0, \quad \forall \xi, y_k \in R, \tag{14}$$

z_{km} is the electrical conductivity tensor with common properties of symmetry and ellipticity, λ_{ij} is a symmetrical elliptic heat conductivity tensor, ρ, c_v and T_0 represent a mass density, specific heat and a fixed uniform reference temperature of a piezoelectric specimen, respectively. Here and everywhere below the ordinary summation by repetitive indices is expected.

To take into account a viscosity effect in pyroelectric materials, we modify the constitutive equation (4) for stress σ_{ij} by adding the term proportional to strain velocity. Therefore, the stress-relation now looks in the following way:

$$\sigma_{ij} = c_{ijkm}[\varepsilon_{km} - \alpha_{km}\theta] - e_{kij}E_k + a_{ijkm}\varepsilon'_{km}, \tag{15}$$

where a_{ijkm} is a viscosity coefficients tensor with common properties of symmetry and ellipticity.

To characterize the interaction of piezoelectric specimen with the environment, we must consider the boundary conditions. The boundary conditions for mechanical and heat fields are:

$$\begin{cases} u_i = 0 \quad on \quad \Gamma_u \times [0,T], \ \Gamma_u \subset \Gamma, mes(\Gamma_u) > 0, \\ \sigma_{ij}n_j = \hat{\sigma}_i \quad on \quad \Gamma_\sigma \times [0,T], \Gamma_\sigma := \Gamma \setminus \Gamma_u, \end{cases} \tag{16}$$

$$\begin{cases} \theta = 0 \quad on \quad \Gamma_\theta \times [0,T], \ \Gamma_\theta \subset \Gamma, mes(\Gamma_\theta) > 0, \\ q_i n_i = \hat{q} \quad on \quad \Gamma_q \times [0,T], \Gamma_q := \Gamma \setminus \Gamma_\theta. \end{cases} \tag{17}$$

Note that nonuniform boundary conditions on parts Γ_u and Γ_θ can be always transformed into uniform ones.

Similarly, the boundary conditions at the interface between the pyroelectric specimen and an ideal dielectric can be described in the following way:

$$\left[D_k' + J_k\right] n_k = 0 \quad \text{on} \;\; \Gamma_d, \; \Gamma_d \subset \Gamma. \tag{18}$$

Many pyroelectric materials and devices are operated under high electric field, which is applied through surface electrodes. We suppose that the electrode has a constant electric potential p_e on its surface, and is soft enough, so that it does not transfer any mechanical loadings. In this case we consider the following boundary conditions

$$p = 0 \quad on \quad \Gamma_p \times [0, T], \; \Gamma_p \subset \Gamma, mes(\Gamma_p) > 0 \quad \text{(grounded electrode)}, \tag{19}$$

and

$$\begin{cases} \int\limits_{\Gamma_e} \left[D_k' + J_k\right] n_k d\gamma = I, \\ p = const \quad \text{on} \;\; \Gamma_e, \quad \Gamma_e = \Gamma \setminus (\Gamma_d \cap \Gamma_p), \end{cases} \tag{20}$$

where I defines the external electrical current.

In order to terminate the formulation of initial boundary value problem of classic piezothermoelectricity, we consider the initial conditions

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u_0}, \quad \boldsymbol{u'}|_{t=0} = \boldsymbol{v_0}, p|_{t=0} = p_0, \quad \theta|_{t=0} = \theta_0 \quad in \;\; \Omega. \tag{21}$$

The aforementioned mathematical model of thermopiezoelectricity was considered in [16, 17], where its well-posedness is proved. Also a finite element based numerical scheme for solving this problem was constructed and the results of numerical experiments are described in [4, 18].

In present work, instead of (8), we use modified Fourier‘s law (also known as Maxwell-Cattaneo equation):

$$\tau q_i' + q_i = -\lambda_{ij}\theta_{,j}. \tag{22}$$

Here the parameter $\tau > 0$ is so-called "relaxation time". This assumption ensures finite speeds of heat wave propagation and was firstly introduced by Lord and Shulman in [10] and was firstly applied to thermopiezoelectricity theory by Chandrasekharaiah in [5]. Also, for convenience, similar to how Chandrasekharaiah did in [5], we introduce artificial coefficients b_{ij} in the way that the following condition is held:

$$T_0 b_{ij}\lambda_{jm} = \delta_{im}, \text{ where } \delta_{im} \text{ are the elements of the unit matrix,} \tag{23}$$

and they satisfy ellipticity conditions:

$$b_{ij} y_i y_j \geq 0 \quad \forall y_i, y_j \in R. \tag{24}$$

Then the modified Fourier‘s law can be rewritten in the following form:

$$\tau b_{ij} q_i' + b_{ij} q_i = -T_0^{-1}\theta_{,j}. \tag{25}$$

Using Maxwell-Cattaneo equation (22) implies, that for Lord-Shulman theory a heat flux $\boldsymbol{q}$ is an additional independent variable. Therefore, the initial conditions (21) must be rewritten into:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u_0}, \quad \boldsymbol{u'}|_{t=0} = \boldsymbol{v_0}, p|_{t=0} = p_0, \quad \theta|_{t=0} = \theta_0, \quad \boldsymbol{q}|_{t=0} = \boldsymbol{q_0} \quad in \;\; \Omega. \tag{26}$$

Thus, the equations (1)-(3), (5)-(7), (9), (15) and (25) together with boundary conditions (16)-(20) and initial conditions (26) define the Lord-Shulman mathematical model of thermopiezoelectricity (initial boundary value problem of LS-thermopiezo-electricity).

3. Variational problem

Let us introduce the spaces of admissible elactic displacements, electric potentials, temperature increments and heat fluxes respectively:

$$\begin{array}{l} V = \left\{ \boldsymbol{v} \in [H^1(\Omega)]^d | \boldsymbol{v} = 0 \ \text{ on } \ \Gamma_u \right\}, \\ X = \left\{ \xi \in H^1(\Omega) | \xi = 0 \ \text{ on } \ \Gamma_p, \ \ \xi = const \ \text{ on } \ \Gamma_e \right\} \\ Y = \left\{ \eta \in H^1(\Omega) | \eta = 0 \ \text{ on } \ \Gamma_\theta \right\}, \\ Z = \left\{ \boldsymbol{\zeta} \in [L^2(\Omega)]^d \right\}, \end{array} \tag{27}$$

and notations

$$\Phi = V \times X \times Y \times Z, \ \Phi_1 = V \times X \times Y, \ G = L^2(\Omega), \ H = G^d. \tag{28}$$

Here symbol $H^m(\Omega)$ means a standard Sobolev space.

After applying the principle of virtual works to initial boundary value problem of LS-thermopiezoelectricity, we obtain the following variational problem:

$$\left\{ \begin{array}{l} \textit{given } \psi_0 = (\boldsymbol{u_0}, p_0, \theta_0, \boldsymbol{q_0}) \in \Phi, \ \boldsymbol{v_0} \in H \textit{ and } (l, r, \mu) \in L^2(0,T;\Phi'); \\ \textit{find } \ \psi = (\boldsymbol{u}, p, \theta, \boldsymbol{q}) \in L^2(0,T;\Phi) \textit{ such that} \\ m(\boldsymbol{u}''(t), \boldsymbol{v}) + a(\boldsymbol{u}'(t), \boldsymbol{v}) + c(\boldsymbol{u}(t), \boldsymbol{v}) - e(p(t), \boldsymbol{v}) - \\ \qquad -\gamma(\theta(t), \boldsymbol{v}) = < l(t), \boldsymbol{v} >, \\ \chi(p'(t), \xi) + e(\xi, \boldsymbol{u}'(t)) + z(p(t), \xi) + \pi(\theta'(t), \xi) = < r(t), \xi >, \\ s(\theta'(t), \eta) + \pi(\eta, p'(t)) + \gamma(\eta, \boldsymbol{u}'(t)) - g(\boldsymbol{q}(t), \eta) = < \mu(t), \eta >, \\ \tau b(\boldsymbol{q}'(t), \boldsymbol{\zeta}) + b(\boldsymbol{q}(t), \boldsymbol{\zeta}) + g(\boldsymbol{\zeta}, \theta(t)) = 0 \ \ \forall \ t \in (0,T], \\ m(\boldsymbol{u}'(0) - \boldsymbol{v_0}, \boldsymbol{v}) = 0, \qquad c(\boldsymbol{u}(0) - \boldsymbol{u_0}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in V, \\ \chi(p(0) - p_0, \xi) = 0 \qquad \forall \xi \in X, \\ s(\theta(0) - \theta_0, \eta) = 0 \qquad \forall \eta \in Y, \\ b(\boldsymbol{q}(0) - \boldsymbol{q_0}, \boldsymbol{\zeta}) = 0 \qquad \forall \boldsymbol{\zeta} \in Z \end{array} \right. \tag{29}$$

The introduced bilinear and linear forms are as follows:

$$\begin{array}{l} m(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega \rho u_i v_i dx = \int\limits_\Omega \rho \boldsymbol{u}.\boldsymbol{v} dx, \ \ a(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega a_{ijkm} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{km}(\boldsymbol{v}) dx, \\ c(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega c_{ijkm} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{km}(\boldsymbol{v}) dx, \ \ < l, \boldsymbol{v} > := \int\limits_\Omega \rho f_i v_i dx + \int\limits_{\Gamma_\sigma} \hat{\sigma}_i v_i d\gamma, \\ \gamma(\xi, \boldsymbol{v}) := \int\limits_\Omega \xi c_{ijkm} \alpha_{km} \varepsilon_{ij} \boldsymbol{v}) dx, \\ e(\xi, \boldsymbol{v}) := \int\limits_\Omega e_{kij} E_k(\xi) \varepsilon_{ij}(\boldsymbol{v}) dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \\ \chi(p, \xi) := \int\limits_\Omega \chi_{km} E_k(p) E_m(\xi) dx, \ \ z(p, \xi) := \int\limits_\Omega z_{km} E_k(p) E_m(\xi) dx, \\ < r, \xi > := I \xi|_{\Gamma_e} \quad \forall p, \xi \in X, \\ \pi(\eta, \xi) = \int\limits_\Omega \eta \pi_k E_k(\xi) dx, \ \ s(\theta, \eta) = \int\limits_\Omega \rho c_v T_0^{-1} \theta \eta dx, \\ < \mu, \eta > := \int\limits_\Omega T_0^{-1} \rho w \eta dx - \int\limits_{\Gamma_h} T_0^{-1} \hat{h} \eta d\gamma \quad \forall \eta, \theta \in Y, \\ b(\boldsymbol{q}, \boldsymbol{\zeta}) = \int\limits_\Omega b_{ij} q_i \zeta_j dx, \ \ g(\boldsymbol{\zeta}, \eta) = \int\limits_\Omega T_0^{-1} \zeta_k \eta_{,k} dx \quad \forall \boldsymbol{q}, \boldsymbol{\zeta} \in Z. \end{array} \tag{30}$$

Now suppose the harmonic loadings with angular frequency $\omega > 0$ are applied to the piezoelectric specimen:

$$\begin{array}{l} l(t) = (l_1 + il_2)e^{-i\omega t}, \\ r(t) = (r_1 + ir_2)e^{-i\omega t}, \\ \mu(t) = (\mu_1 + i\mu_2)e^{-i\omega t}, \quad \forall\, t \in (0, T]. \end{array} \tag{31}$$

Then we can look for approximate solutions of problem (29) in the form of the following expansions:

$$\begin{array}{l} \boldsymbol{u}(x,t) \cong (\boldsymbol{u_1}(x) + i\boldsymbol{u_2}(x))e^{-i\omega t}, \\ p(x,t) \cong (p_1(x) + ip_2(x))e^{-i\omega t}, \\ \theta(x,t) \cong (\theta_1(x) + i\theta_2(x))e^{-i\omega t}, \\ \boldsymbol{q}(x,t) \cong (\boldsymbol{q_1}(x) + i\boldsymbol{q_2}(x))e^{-i\omega t}, \end{array} \tag{32}$$

where $\boldsymbol{u_1}(x)$, $\boldsymbol{u_2}(x)$, $p_1(x)$, $p_2(x)$, $\theta_1(x)$, $\theta_2(x)$ and $\boldsymbol{q_1}(x)$, $\boldsymbol{q_2}(x)$ are the unknown amplitudes of mechanical displacement, electric potential, temperature increment and heat flux respectively.

After substitution of (31) and (32) into (29) and neglection of its initial conditions, we obtain the variational problem for forced harmonic vibrations of piezoelectric specimen:

$$\left\{ \begin{array}{l} \textit{given}\ \omega > 0,\ (l_1, l_2, r_1, r_2, \mu_1, \mu_2, 0, 0) \in W' = \Phi' \times \Phi'; \\ \textit{find}\ \ \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W = \Phi \times \Phi\ \textit{such that} \\ -\omega^2 m(\boldsymbol{u_1}, \boldsymbol{v_2}) + \omega a(\boldsymbol{u_2}, \boldsymbol{v_2}) + c(\boldsymbol{u_1}, \boldsymbol{v_2}) - e(p_1, \boldsymbol{v_2}) - \\ \qquad -\gamma(\theta_1, \boldsymbol{v_2}) = < l_1, \boldsymbol{v_2} >, \\ -\omega^2 m(\boldsymbol{u_2}, \boldsymbol{v_1}) - \omega a(\boldsymbol{u_1}, \boldsymbol{v_1}) + c(\boldsymbol{u_2}, \boldsymbol{v_1}) - e(p_2, \boldsymbol{v_1}) - \\ \qquad -\gamma(\theta_2, \boldsymbol{v_1}) = < l_2, \boldsymbol{v_1} >, \\ \omega\chi(p_2, \xi_1) + \omega e(\xi_1, \boldsymbol{u_2}) + z(p_1, \xi_1) + \omega\pi(\theta_2, \xi_1) = < r_1, \xi_1 >, \\ -\omega\chi(p_1, \xi_2) - \omega e(\xi_2, \boldsymbol{u_1}) + z(p_2, \xi_2) - \omega\pi(\theta_1, \xi_2) = < r_2, \xi_2 >, \\ \omega s(\theta_2, \eta_1) + \omega\pi(\eta_1, p_2) + \omega\gamma(\eta_1, \boldsymbol{u_2}) - g(\boldsymbol{q_1}, \eta_1) = < \mu_1, \eta_1 >, \\ -\omega s(\theta_1, \eta_2) - \omega\pi(\eta_2, p_1) - \omega\gamma(\eta_2, \boldsymbol{u_1}) - g(\boldsymbol{q_2}, \eta_2) = < \mu_2, \eta_2 >, \\ \omega\tau b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + g(\boldsymbol{\zeta_1}, \theta_1) = 0, \\ -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{\zeta_2}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2}) + g(\boldsymbol{\zeta_2}, \theta_2) = 0 \\ \qquad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \right. \tag{33}$$

Having added all the equations of the problem (33), we introduce the bilinear form $\Pi_\omega : W \times W \to R$ and linear form $\chi_\omega : W \to R$ in the following way:

$$\begin{array}{l} \Pi_\omega(\psi, w) = -\omega^2[m(\boldsymbol{u_1}, \boldsymbol{v_2}) - m(\boldsymbol{u_2}, \boldsymbol{v_1})] + \\ +\omega[a(\boldsymbol{u_1}, \boldsymbol{v_1}) + a(\boldsymbol{u_2}, \boldsymbol{v_2})] + [c(\boldsymbol{u_1}, \boldsymbol{v_2}) - c(\boldsymbol{u_2}, \boldsymbol{v_1})] + \\ +[e(p_2, \boldsymbol{v_1}) - e(p_1, \boldsymbol{v_2}) + e(\xi_1, \boldsymbol{u_2}) - e(\xi_2, \boldsymbol{u_1})] + \\ +[\gamma(\theta_2, \boldsymbol{v_1}) - \gamma(\theta_1, \boldsymbol{v_2}) + \gamma(\eta_1, \boldsymbol{u_2}) - \gamma(\eta_2, \boldsymbol{u_1})] + \\ +[\pi(\theta_2, \xi_1) - \pi(\theta_1, \xi_2) + \pi(\eta_1, p_2) - \pi(\eta_2, p_1)] + \\ +[\chi(p_2, \xi_1) - \chi(p_1, \xi_2)] + \omega^{-1}[z(p_1, \xi_1) + z(p_2, \xi_2)] + \\ +[s(\theta_2, \eta_1) - s(\theta_1, \eta_2)] + \\ +\omega^{-1}[g(\boldsymbol{\zeta_1}, \theta_1) + g(\boldsymbol{\zeta_2}, \theta_2) - g(\boldsymbol{q_1}, \eta_1) - g(\boldsymbol{q_2}, \eta_2)] + \\ +\tau[b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) - b(\boldsymbol{q_1}, \boldsymbol{\zeta_2})] + \omega^{-1}[b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2})] \\ \qquad \forall\ \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W, \\ \qquad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \tag{34}$$

$$\begin{aligned} &< \chi_\omega, w >= - < l_2, \boldsymbol{v_1} > + \omega^{-1}[< r_1, \xi_1 > + < \mu_1, \eta_1 >] + \\ &+ < l_1, \boldsymbol{v_2} > + \omega^{-1}[< r_2, \xi_2 > + < \mu_2, \eta_2 >] \\ &\qquad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{aligned} \tag{35}$$

Then variational problem for forced harmonic vibrations of pyroelectric can be rewritten as follows:

$$\begin{cases} \textit{given}\ \omega > 0,\ \chi_\omega \in W' = \Phi' \times \Phi'; \\ \textit{find}\ \ \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W = \Phi \times \Phi\ \textit{such that} \\ \Pi_\omega(\psi, w) =< \chi_\omega, w > \quad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2}) \in W. \end{cases} \tag{36}$$

4. Well-posedness of the variational problem

Theorem 1. *Let us define the bilinear form $k(\cdot,\cdot)$ as follows:*

$$k(\theta, \eta) = \int_\Omega T_0^{-1} \boldsymbol{\Lambda} \nabla\theta \nabla\eta dx, \tag{37}$$

where $\boldsymbol{\Lambda} = \{\lambda_{ij}\}$ is matrix of thermal conductivity coefficients. Then the below equality is held:

$$(1 + \omega^2\tau^2)[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = k(\theta_1, \theta_1) + k(\theta_2, \theta_2), \tag{38}$$

where $\boldsymbol{q_1}, \boldsymbol{q_2}, \theta_1, \theta_2$ are the solutions of variational problems (33) and (36), defining amplitudes of heat flux and temperature increment correspondingly.

Proof.

The modified Fourier law

$$\tau \boldsymbol{q}' + \boldsymbol{q} = -\boldsymbol{\Lambda}\nabla\theta \tag{39}$$

is rewritten for the case of harmonic vibrations:

$$-i\omega\tau(\boldsymbol{q_1} + i\boldsymbol{q_2})e^{-iwt} + (\boldsymbol{q_1} + i\boldsymbol{q_2})e^{-iwt} = -\boldsymbol{\Lambda}(\nabla\theta_1 + i\nabla\theta_2)e^{-iwt}. \tag{40}$$

The expression (40) is then splitted into real and imaginary parts. As a result, we obtain:

$$\begin{aligned} \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2} &= -\boldsymbol{\Lambda}\nabla\theta_1, \\ \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1} &= -\boldsymbol{\Lambda}\nabla\theta_2. \end{aligned} \tag{41}$$

After multiplying equations of (41) by $T_0^{-1}\nabla\theta_1$ and $T_0^{-1}\nabla\theta_2$ respectively and integration over the domain Ω we get:

$$\begin{aligned} g(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \theta_1) &= -k(\theta_1, \theta_1), \\ g(\boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}, \theta_2) &= -k(\theta_2, \theta_2). \end{aligned} \tag{42}$$

Then two last equations of the variational problem (33) are considered and a substitution of admissible functions $\boldsymbol{\zeta_1} = \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}$ and $\boldsymbol{\zeta_2} = \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}$ is performed respectively:

$$\begin{aligned} \omega\tau b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + g(\boldsymbol{\zeta_1}, \theta_1) = 0,\ \boldsymbol{\zeta_1} = \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \\ -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{\zeta_2}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2}) + g(\boldsymbol{\zeta_2}, \theta_2) = 0,\ \boldsymbol{\zeta_2} = \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}. \end{aligned} \tag{43}$$

After simplifying the first equation of (43) with taking into account the relations (42) we obtain:

$$\begin{aligned} &\omega\tau b(\boldsymbol{q_2}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) + b(\boldsymbol{q_1}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) + g(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \theta_1) = 0, \\ &b(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) = k(\theta_1, \theta_1). \end{aligned} \tag{44}$$

Similarly, simplifying the second equation of (43) with taking into account the relations (42) we get:

$$\begin{array}{l} -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}) + g(\boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}, \theta_2) = 0, \\ b(\boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}, \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}) = k(\theta_2, \theta_2). \end{array} \tag{45}$$

The last equations of the relations (44) and (45) can be rewritten in the following way:

$$\begin{array}{l} b(\boldsymbol{q_1}, \boldsymbol{q_1}) + 2\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2}) + \omega^2\tau^2 b(\boldsymbol{q_2}, \boldsymbol{q_2}) = k(\theta_1, \theta_1), \\ \omega^2\tau^2 b(\boldsymbol{q_1}, \boldsymbol{q_1}) - 2\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2}) + b(\boldsymbol{q_2}, \boldsymbol{q_2}) = k(\theta_2, \theta_2). \end{array} \tag{46}$$

After summarizing these 2 equations of (46) we obtain:

$$(1 + \omega^2\tau^2)[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = k(\theta_1, \theta_1) + k(\theta_2, \theta_2). \tag{47}$$

□

Let us introduce a scalar product on the space W in the following way:

$$\begin{array}{l} ((y, w)) = \sum\limits_{i=1}^{2} [a(\boldsymbol{u_i}, \boldsymbol{v_i}) + z(p_i, \xi_i) + \frac{1}{2} b(\boldsymbol{q_i}, \boldsymbol{\zeta_i}) + \frac{1}{2(1+\omega^2\tau^2)} k(\theta_i, \eta_i)] \\ \quad \forall\ y = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W, \\ \quad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \tag{48}$$

We also introduce a norm generated by the scalar product (48):

$$|||y|||^2 = (y, y)\ \forall y \in W. \tag{49}$$

Then the following estimations are easy noticed:

$$\begin{array}{l} |\Pi_\omega(y, w)| \leq M_1(\omega)|||y||| \cdot |||w|||, \\ M_1(\omega) = C\ max\{\omega^{-1}, 1, \omega, \omega^2\},\ \forall y, w \in W, \end{array} \tag{50}$$

and

$$\begin{array}{l} | < \chi_\omega, w > | \leq M_2(\omega)||\chi_\omega||_* \cdot |||w|||, \\ M_2(\omega) = C\ max\{\omega^{-1}, 1\},\ \forall w \in W. \end{array} \tag{51}$$

Here and everywhere the symbol C means a positive constant value, which is not dependent on solutions of variational problem (36).

Consider now the expression for $\Pi_\omega(w, w)$:

$$\begin{array}{l} \Pi_\omega(w, w) = -\omega^2[m(\boldsymbol{u_1}, \boldsymbol{u_2}) - m(\boldsymbol{u_2}, \boldsymbol{u_1})]+ \\ +\omega[a(\boldsymbol{u_1}, \boldsymbol{u_1}) + a(\boldsymbol{u_2}, \boldsymbol{u_2})] + [c(\boldsymbol{u_1}, \boldsymbol{u_2}) - c(\boldsymbol{u_2}, \boldsymbol{u_1})]+ \\ +[e(p_2, \boldsymbol{u_1}) - e(p_1, \boldsymbol{u_2}) + e(p_1, \boldsymbol{u_2}) - e(p_2, \boldsymbol{u_1})]+ \\ +[\gamma(\theta_2, \boldsymbol{u_1}) - \gamma(\theta_1, \boldsymbol{u_2}) + \gamma(\theta_1, \boldsymbol{u_2}) - \gamma(\theta_2, \boldsymbol{u_1})]+ \\ +[\pi(\theta_2, p_1) - \pi(\theta_1, p_2) + \pi(\theta_1, p_2) - \pi(\theta_2, p_1)]+ \\ +[\chi(p_2, p_1) - \chi(p_1, p_2)]+ \\ +\omega^{-1}[z(p_1, p_1) + z(p_2, p_2)] + [s(\theta_2, \theta_1) - s(\theta_1, \theta_2)]+ \\ +\omega^{-1}[g(\boldsymbol{q_1}, \theta_1) + g(\boldsymbol{q_2}, \theta_2) - g(\boldsymbol{q_1}, \theta_1) - g(\boldsymbol{q_2}, \theta_2)]+ \\ +\tau[b(\boldsymbol{q_2}, \boldsymbol{q_1}) - b(\boldsymbol{q_1}, \boldsymbol{q_2})] + \omega^{-1}[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = \\ = \sum\limits_{i=1}^{2} [\omega a(\boldsymbol{u_i}, \boldsymbol{u_i}) + \omega^{-1} z(p_i, p_i) + \omega^{-1} b(\boldsymbol{q_i}, \boldsymbol{q_i})] = \\ = \sum\limits_{i=1}^{2} [\omega a(\boldsymbol{u_i}, \boldsymbol{u_i}) + \omega^{-1} z(p_i, p_i)+ \\ +\omega^{-1}(\frac{1}{2} b(\boldsymbol{q_i}, \boldsymbol{q_i}) + \frac{1}{2(1+\omega^2\tau^2)} k(\theta_i, \theta_i))] \geq \alpha(\omega) \cdot |||w|||^2, \end{array} \tag{52}$$

where $\alpha(\omega) = min\{\omega^{-1}, \omega\} \quad \forall w \in W.$

Since the statements (50 - 52) are held and they are actually the conditions of Lions-Lax-Milgram theorem, the following theorem is then proved:

Theorem 2. *For each $w > 0$ and $\tau > 0$ variational problem (36) has a unique solution $\psi \in W$, which satisfy the relation:*

$$|||\psi||| \leq \alpha^{-1}(\omega) M_2(\omega) ||\chi_\omega||_*. \tag{53}$$

5. Galerkin discretization

Galerkin scheme makes a transition of the solution of variational problem (33) from space $W := \Phi \times \Phi$ to its finite-dimensional subspace $W_h := \Phi_h \times \Phi_h$, $\Phi_h \subset \Phi$, $\dim W_h = N(h) < +\infty$. Thus, discretized variational problem (36) looks in the following way:

$$\left\{ \begin{array}{l} \textit{given angular frequency} \;\; \omega > 0, \;\; \chi_\omega \in W', \\ \textit{approximations space} \;\; W_h \subset W, \; \dim W_h < +\infty; \\ \textit{find vector} \;\; \psi_h = (\boldsymbol{u_{1h}}, \boldsymbol{u_{2h}}, p_{1h}, p_{2h}, \theta_{1h}, \theta_{2h}, \boldsymbol{q_{1h}}, \boldsymbol{q_{2h}}) \in W_h \\ \textit{such that} \;\; \Pi_\omega(\psi_h, \varphi) = < \chi_\omega, \varphi > \quad \forall\, \varphi \in W_h. \end{array} \right. \tag{54}$$

Since problem (36) is well-posed, the same applies to its discretized counterpart (54).

In the space W we select some basis functions $\{w_i\}_{i=1}^\infty$. For each natural number $m \geq 1$, $h = 1/m$ a sequence of approximation spaces W_h and operators of orthogonal projection $\mathrm{Pr}_h : W \to W_h$ are defined so that a set $\{w_i\}_{i=1}^m$ is a base of W_h, $((\psi - \mathrm{Pr}_h\, \psi, w)) = 0 \;\forall\, \psi \in W, \;\forall\, w_h \in W_h$.

Now variational problem (36) is replaced by a sequence of the following problems:

$$\left\{ \begin{array}{l} \textit{given } \omega > 0, \; \chi_\omega \in W' \textit{and } h > 0, \; W_h \subset W, \; \dim W_h = m < +\infty; \\ \textit{find vector } \psi_h \in W_h \textit{ such that} \\ \Pi_\omega(\psi_h, w) = < \chi_\omega, w > \;\; \forall w \;\in W_h. \end{array} \right. \tag{55}$$

Theorem 3. *Let $\boldsymbol{\psi} \in W$ be a solution of problem (36) with parameter $\omega > 0$. Then a sequence of Galerkin approximations $\{\boldsymbol{\psi_h}\} \subset W$ is unambiguously defined by the solutions of problems (55) and has the following properties:*

$$||\psi - \psi_h||_W \leq \alpha^{-1} M_1(\omega) \inf_{w \in W_h} ||\psi - w||_W \;\; \forall\; h > 0; \tag{56}$$

$$\lim_{h \to 0} ||\psi - \psi_h||_W = 0. \tag{57}$$

Proof. The correctness of the inequality (56) is based on the fact that

$$\Pi_\omega(\psi - \psi_h, w) = 0 \;\forall\; w \in W_h,$$

and the estimation

$$\begin{array}{l} \alpha ||\psi - \psi_h||_W^2 \leq \Pi_\omega(\psi - \psi_h, \psi - \psi_h) = \Pi_\omega(\psi - \psi_h, \psi - w) \leq \\ \leq M_1(\omega) ||\psi - \psi_h||_W ||\psi - w||_W \;\forall\; w \in W_h. \end{array}$$

Taking into account the density of sequence of spaces $\{W_h\}$ in the separable space W

$$\lim_{h \to 0} ||w - Pr_h w||_W = 0 \;\forall\; w \in W.$$

Therefore, basing on the equality

$$\inf_{w\in W_h} ||\psi - w||_W = ||\psi - \Pr_h \psi||_W$$

and (56) we can conclude the correctness of (57), when $\omega > 0$. □

Theorem 4. *on the convergence of FEM approximations.*

Let $\psi \in W$ be a solution of problem (36) and exists a natural number $k \geq 1$ such that $\psi \in W \cap [H^{k+1}(\Omega)]^{2(d+1)}$. Let approximations ψ_h be defined by solving problem (55) in the spaces $W_h \subset W$, which are constructed with making use of piecewise-polynomial functions of FEM and have the following property:

for each $\varphi \in W \cap [H^{k+1}(\Omega)]^{2(d+1)}$, $k \geq 1$ there exist $\varphi_h \in W_h$ and $C = const > 0$ such that $\|\varphi - \varphi_h\|_{m,\Omega} \leq C \cdot h^{k+1-m}\|\varphi\|_{k+1,\Omega}$, $0 \leq m \leq k$, where h is the diameter of finite element mesh and k is the greatest degree of full polynomial of d variables, which is precisely defined by basis functions of W_h on each finite element.

Then the convergence of sequence $\psi_h \subset W$ is characterized by the estimation

$$\|\psi - \psi_h\| \leq C \cdot h^k ||\psi||_{k+1,\Omega}, \tag{58}$$

where $C = const > 0$ is not dependent on values we are looking for.

Proof. The estimation (58) is implied from the inequality (56), the equivalence of norms $||\cdot||_W$ and $||\cdot||_{1,\Omega}$ on W and the density properties defined in the theorem body.

$$\|\psi - \psi_h\|_W \leq \alpha^{-1} M_1(\omega) \inf_{w\in W_h} \|\psi - w\| = \|\psi - w\|_{1,\Omega} \leq C \cdot h^k ||\psi||_{k+1,\Omega}$$

□

Let us now pay a deeper attention to the aforementioned selection of finite-dimensional subspace $W_h \in W$. Taking into account the definition of W_h that is $W_h = V_h \times X_h \times Y_h \times Z_h \times V_h \times X_h \times Y_h \times Z_h$, where

$$\begin{array}{l} V_h \subset V,\ X_h \subset X,\ Y_h \subset Y,\ Z_h \subset Z, \\ dimV_h < +\infty,\ dimX_h < +\infty,\ dimY_h < +\infty,\ dimZ_h < +\infty. \end{array} \tag{59}$$

we can write the expansions of solution amplitudes as following:

$$\begin{array}{l} \boldsymbol{u_{\alpha h}} \simeq \sum\limits_{i=0}^{N} \boldsymbol{U_\alpha} \phi_i^V(x), \\ p_{\alpha h} \simeq \sum\limits_{i=0}^{N} \boldsymbol{P_\alpha} \phi_i^X(x), \\ \theta_{\alpha h} \simeq \sum\limits_{i=0}^{N} \boldsymbol{\Theta_\alpha} \phi_i^Y(x), \\ \boldsymbol{q_{\alpha h}} \simeq \sum\limits_{i=0}^{N} \boldsymbol{Q_\alpha} \phi_i^Z(x), \alpha = 1, 2, \end{array} \tag{60}$$

where $\phi_i^V(x), \phi_i^X(x), \phi_i^Y(x)$ and $\phi_i^Z(x)$ are the basis functions of spaces V, X, Y and Z respectively. Then we obtain the system of linear equations for finding

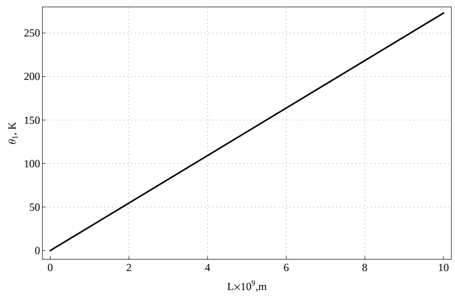

FIG. 1. Amplitude of temperature θ_1

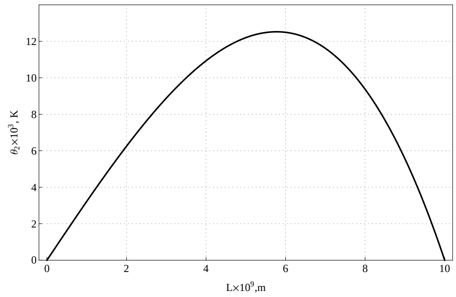

FIG. 2. Amplitude of temperature θ_2

nodal values of the unknown amplitudes:

$$\begin{bmatrix} \omega\boldsymbol{A} & -[-\omega^2\boldsymbol{M}+\boldsymbol{C}] & 0 & \boldsymbol{E^T} & 0 & \boldsymbol{Y^T} & 0 & 0 \\ [-\omega^2\boldsymbol{M}+\boldsymbol{C}] & \omega\boldsymbol{A} & -\boldsymbol{E^T} & 0 & -\boldsymbol{Y^T} & 0 & 0 & 0 \\ 0 & \boldsymbol{E} & \omega^{-1}\boldsymbol{Z} & \omega\boldsymbol{X} & 0 & \boldsymbol{\Pi^T} & 0 & 0 \\ -\boldsymbol{E} & 0 & -\boldsymbol{X} & \omega^{-1}\boldsymbol{Z} & -\boldsymbol{\Pi^T} & 0 & 0 & 0 \\ 0 & \boldsymbol{Y} & 0 & \boldsymbol{\Pi} & 0 & \boldsymbol{S} & -\omega^{-1}\boldsymbol{G^T} & 0 \\ -\boldsymbol{Y} & 0 & -\boldsymbol{\Pi} & 0 & -\boldsymbol{S} & 0 & 0 & -\omega^{-1}\boldsymbol{G^T} \\ 0 & 0 & 0 & 0 & \omega^{-1}\boldsymbol{G} & 0 & \omega^{-1}\boldsymbol{B} & \tau\boldsymbol{B} \\ 0 & 0 & 0 & 0 & 0 & \omega^{-1}\boldsymbol{G} & -\tau\boldsymbol{B} & \omega^{-1}\boldsymbol{B} \end{bmatrix}.$$

$$\begin{aligned} &\cdot [\boldsymbol{U_1}, \boldsymbol{U_2}, \boldsymbol{P_1}, \boldsymbol{P_2}, \boldsymbol{\Theta_1}, \boldsymbol{\Theta_2}, \boldsymbol{Q_1}, \boldsymbol{Q_2}]^T = \\ &= \left[-\boldsymbol{L_2}, \boldsymbol{L_1}, \omega^{-1}\boldsymbol{R_1}, \omega^{-1}\boldsymbol{R_2}, \omega^{-1}\boldsymbol{F_1}, \omega^{-1}\boldsymbol{F_2}, 0, 0\right]^T. \end{aligned} \tag{61}$$

Here the elements of the matrices and vectors are computed using the bilinear and linear forms defined in (30), for example $\boldsymbol{A} = \{a_{ij}\} = \{a(\phi_i^V, \phi_j^V)\}$. The matrix of the system of equations (61) is positively defined, but not the symmetric one. More precisely, it can be represented as the sum of positively defined symmetric matrix and a skew-symmetric one.

6. Numerical experiments

We consider a piezoelectric bar with length $L = 10^{-8}m$ made of PZT-4 ceramics.A harmonic thermal loading with angular frequency $\omega = 3 \cdot 10^6 rad/s$ is applied to the right edge of the bar. So, the boundary conditions for thermal field are:

$$\theta_1(0) = 0K,\ \theta_1(L) = 273K,\ \theta_2(0) = 0K,\ \theta_2(L) = 0K. \tag{62}$$

On the left edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Dirichlet type :

$$u_1(0) = 0m,\ u_2(0) = 0m,\ p_1(0) = 0V,\ p_2(0) = 0V. \tag{63}$$

On the right edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Neumann type :

$$\sigma_1(L) = 0N \cdot m^{-2},\ \sigma_2(L) = 0N \cdot m^{-2},\ J_1(L) = 0A,\ J_2(L) = 0A. \tag{64}$$

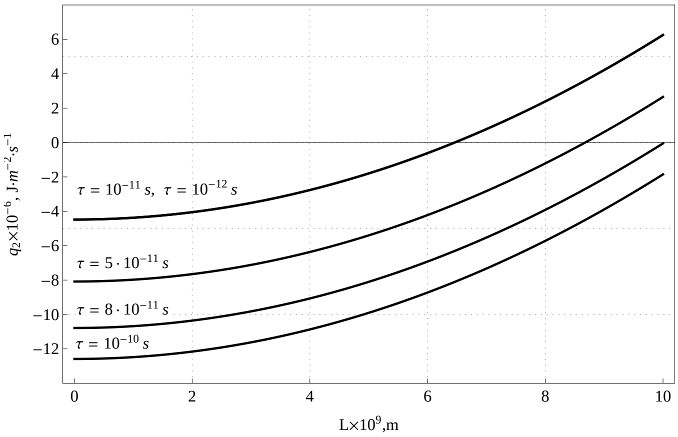

FIG. 3. Amplitude of heat flux component q_2 for the PZT-4 bar for relaxation times $\tau = 10^{-10}, 8 \cdot 10^{-11}, 5 \cdot 10^{-11}, 10^{-11}, 10^{-12}s$

We take the coefficients of PZT-4 as in [20]:

$\rho = 7500[kg/m^3]$
$c_v = 350[J/kg \cdot K]$
$\lambda = 1.1[W/m \cdot K]$
$c = 115 \times 10^9[N/m^2]$
$e = 15.1[C/m^2]$
$\pi = 2.7 \times 10^{-4}[C/K \cdot m^2]$
$\chi = 6.46 \times 10^{-9}[C^2/N \cdot m^2]$
$\alpha = 3.13 \times 10^{-5}[K^{-1}]$

Also we take $z = 5 \times 10^{-12}[\Omega^{-1} \cdot m^{-1}]$, $a = 40[m^2 \cdot s^{-1}]$ and $T_0 = 298[K]$. As mentioned in [20], the value of the relaxation time τ for PZT-4 cannot be found in the literature. However, the relation time τ is determined for different type of materials, ranging from 10^{-10} for gases to 10^{-14} for metals. Therefore, in our numerical experiments we will use the values of relaxation time $\tau = 10^{-10}, 8 \cdot 10^{-11}, 5 \cdot 10^{-11}, 10^{-11}, 10^{-12}s$. For discretization by spatial variable we divide the interval $[0, L]$ into $N = 256$ finite elements with piecewise linear solution approximations on them.

Fig. 1 shows that under these boundary conditions and angular frequency $\omega = 3 \cdot 10^6 rad \cdot s^{-1}$ the calculated temperature increment θ_1 is changing linearly along the bar, regardless of the value of relaxation time τ. Fig. 2 depicts the calculated amplitude θ_2 of the temperature increment. It is also not dependent on the value of relaxation time τ.

On the other hand, as Fig.3 shows, the amplitude of heat flux q_2 is dependent on the parameter τ. It worth mentioning, that the amplitude calculated with $\tau = 10^{-12}s$ is almost identical to the one obtained as a solution of the classical thermopiezoelectricity problem for forced harmonic vibrations (when no modified Fourier law is taken into account).

7. CONCLUSIONS

The harmonic vibrations of the pyroelectric materials have been studied under generalized Lord-Shulman thermopiezoelectricity theory. The variational problem for this special case has been formulated and its well-posedness has been proved. Then the discretization of the problem using Galerkin-method has been performed. The finite element method has been utilized to construct the bases of approximation spaces of the discretized problem. The rate of convergence of FEM-approximations has been determined. After the discretization we obtain the system of linear algebraic equations with positively defined matrix in its left part. Therefore, we can be sure that the solution of that system always exists. The numerical experiment of applying a harmonic thermal loading to the pyroelectric bar has been set up and studied. The results of the experiment showed the significant influence of the "relaxation time" parameter on the nodal values of solution amplitudes.

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BOUNDARY VALUE PROBLEM FOR THE TWO-DIMENSIONAL LAPLACE EQUATION WITH TRANSMISSION CONDITION ON THIN INCLUSION

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Резюме. Розглянуто задачу для рівняння Лапласа в обмеженій двовимірній Ліпшицевій області з тонким включенням, на якому задана трансмісійна гранична умова, тобто умова, що містить як стрибок нормальної похідної, так і граничне значення шуканої функції. Доведено еквівалентність задачі у диференціальному формулюванні та відповідної варіаційної задачі. Досліджено питання існування та єдиності розв'язку поставленої задачі у відповідних функціональних просторах. На основі інтегрального подання розв'язку вихідна диференціальна задача зведена до системи граничних інтегральних рівнянь. Побудовано алгоритм чисельного розв'язування отриманої системи інтегральних рівнянь методом колокації. Представлено чисельні результати наближеного розв'язуваня деяких конкретних граничних задач.

Abstract. We consider boundary value problem for Laplace equation in bounded two-dimensional Lipschitz domain with thin inclusion. Transmission boundary condition upon it consists of the jump of normal derivative and the meaning of boundary value of seeking function. We prove the equivalence of initial boundary value problem and connected variational problem. As a result we obtain existence and uniqueness of solution of the posed problem in appropriate functional spaces. Based on the integral representation formula the considered boundary value problem is reduced to the system of boundary integral equations. We construct the algorithm of numerical solution of obtained system by collocation method. Our approach is illustrated by some numerical examples.

The numerical results show that the proposed methods give a good accuracy of reconstructions with an economical computational cost.

1. Introduction

Boundary value problems for the second order elliptic equations with transmission boundary conditions in nonsmooth domains are important class of boundary value problems and were considered by many authors [1]- [4], [7, 8].

We consider a special case of the transmission conditions when they are posed on an open Lipschitz curve. From the mathematical point of view such kind of problem describes stationary temperature field in domain with thin inclusion when the temperature passing through this inclusion is continuous and the heat flux is discontinuous and proportional to the boundary value of temperature.

Key words. Laplace equation; transmission condition; variational problem; open curve.

In oder to obtain convenient mathematical model for this physical problem it's useful to present thin objects as inclusion or crack like an open curve. As a result we get essentially unregular domain and need to introduce corresponding trace maps and appropriate functional spaces [1, 6].

In present paper we use a variational formulation of the posed boundary value problem with transmission condition which gives us opportunity to obtain the existence and uniqueness of solution.

2. Functional spaces and trace operators

Let $\Omega_+ \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain. This means that its boundary curve Σ is locally the graph of a Lipschitz function [5, 6]. Let us note that Σ can be piecewise smooth and have corner points. $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$. We suppose that S is an open Lipschitz curve with the end points c_1 and c_2, $\overline{S} = S \cup \{c_1, c_2\}$ and $\overline{S} \subset \Omega_+$. We denote $\Omega = \Omega_+ \setminus \overline{S}$ and consider S as a part of a some closed bounded Lipschitz curve $\Sigma_0 = \overline{S} \cup S_0$, $\Sigma_0 \subset \Omega_+$.

Since Σ and S are Lipschitz almost everywhere we can define outward pointing vector of the normal $\vec{n}_x$, $x \in \Sigma$ or $x \in S$. Depend on the direction of $\vec{n}_x$, $x \in S$, we consider S as a double sided curve with sides S_+ and S_-.

In Ω_+ we consider the Laplace operator

$$Lu = -\Delta u = -\sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2$$

and connected bilinear form

$$a(u, v) = (\nabla u, \nabla v)_{L_2(\Omega_+)} = \int_{\Omega_+} \left\{ \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right\} dx.$$

We use the Hilbert spaces $H^1(\Omega_+)$ and $H^1(\Omega_+, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega_+)} = \int_{\Omega_+} \left\{ |\nabla u|^2 + u^2 \right\} dx, \quad (u, v)_{H^1(\Omega_+)} = \int_{\Omega_+} \{ (\nabla u, \nabla v) + uv \} \, dx,$$

$$\|u\|^2_{H^1(\Omega_+, L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

$$(u, v)_{H^1(\Omega_+, L)} = (u, v)_{H^1(\Omega_+)} + (Lu, Lv)_{L_2(\Omega_+)}.$$

The trace operators $\gamma^+_{0,\Sigma} : H^1(\Omega_+) \to H^{1/2}(\Sigma)$ and $\gamma^+_{1,\Sigma} : H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ are continuous and surjective [5, 6]. Here $\gamma^+_{1,\Sigma} u \in H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'$ and coincides with $\frac{\partial u}{\partial n_x}$ for $u \in C^1(\overline{\Omega}_+)$.

Let us denote by $C_0^\infty(\Omega)$ the class of infinitely differentiable functions with compact support in Ω. We introduce the Hilbert spaces $H^1(\Omega)$ and $H^1(\Omega, L)$ of real functions with norms

$$\|u\|^2_{H^1(\Omega)} = \int_{\Omega} \left\{ |\nabla u|^2 + u^2 \right\} dx, \tag{1}$$

$$\|u\|^2_{H^1(\Omega, L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)},$$

where derivatives $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$ are defined as

$$\left(\frac{\partial u}{\partial x_i}, \varphi\right)_{L_2(\Omega)} = -\int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\left(u, \frac{\partial \varphi}{\partial x_i}\right)_{L_2(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$.

We consider some trace maps in Ω. We denote $\gamma_{0,S}^{\pm}$ and $\gamma_{1,S}^{\pm}$ the restrictions of trace maps $\gamma_{0,\Sigma_0}^{\pm}$ and $\gamma_{1,\Sigma_0}^{\pm}$ on S respectively [9]. Then we have $\gamma_{0,S}^{\pm} : H^1(\Omega) \to H^{1/2}(S)$ and $\gamma_{1,S}^{\pm} : H^1(\Omega, L) \to H^{-1/2}(S)$.

We introduce the space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma_{0,S}^{\pm}u = 0, \gamma_{0,\Sigma}^{+}u = 0\}$$

and denote dual space $H^{-1}(\Omega) = (H_0^1(\Omega))'$. We also have that $H_0^1(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the norm (1).

In what follows we use the next trace maps: $[\gamma_{0,S}] = \gamma_{0,S}^{+} - \gamma_{0,S}^{-}$, $[\gamma_{1,S}] = \gamma_{1,S}^{+} - \gamma_{1,S}^{-}$. Analogously as it was obtained in [9, 10] for $\mathbb{R}^3$ we can show that

$$[\gamma_{0,S}] : H^1(\Omega) \to H_{00}^{1/2}(S), \quad [\gamma_{1,S}] : H^1(\Omega, L) \to H_{00}^{-1/2}(S),$$

where $H_{00}^{1/2}(S) = \{g \in H^{1/2}(S) : p_0 g \in H^{1/2}(\Sigma_0)\}$. Here $p_0 g$ is extension by zero of the function g on S_0. The norm in $H_{00}^{1/2}(S)$ is given as

$$\|g\|_{H_{00}^{1/2}(S)} = \|p_0 g\|_{H^{1/2}(\Sigma_0)}.$$

$H_{00}^{-1/2}(S) = (H^{1/2}(S))'$, $H^{-1/2}(S) = (H_{00}^{1/2}(S))'$.

We have the first Green's formula for bounded domain with an open curve which in presented case for $u \in H^1(\Omega, L)$ and $v \in H^1(\Omega)$ has the following form:

$$a(u, v) = (Lu, v)_{L_2(\Omega)} + \langle\gamma_{1,S}^{+}u, [\gamma_{0,S}]v\rangle + \langle[\gamma_{1,S}]u, \gamma_{0,S}^{-}v\rangle + \langle\gamma_{1,\Sigma}^{+}u, \gamma_{0,\Sigma}^{+}v\rangle. \quad (2)$$

Here $\langle\cdot,\cdot\rangle$ are relations of duality between $H_{00}^{1/2}(S)$ and $H^{-1/2}(S)$, $H^{1/2}(S)$ and $H_{00}^{-1/2}(S)$, $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ respectively.

Let $\overline{\Omega}_1 \subset \Omega_+$ be a Lipschitz domain bounded by the closed curve Σ_0. $\overline{\Omega}_1 = \Omega_1 \cup \Sigma_0$, $\Omega_2 = \Omega_+ \setminus \overline{\Omega}_1$. We denote by u_i the restriction of $u \in H^1(\Omega)$ to Ω_i, $i = 1, 2$. It's obviously that $u_i \in H^1(\Omega_i)$, $i = 1, 2$.

Lemma 1. *The trace map $\gamma_{0,S}^{-} : H^1(\Omega_+) \to H^{1/2}(S)$ is continuous and surjective.*

Proof. Let $g \in H^{1/2}(S)$ be an arbitrary function. We denote by $pg \in H^{1/2}(\Sigma_0)$ the extension of g on Σ_0. The trace map $\gamma_{0,\Sigma_0}^{-} : H^1(\Omega_1) \to H^{1/2}(\Sigma_0)$ is continuous and surjective. Thus there exists function $u_1 \in H^1(\Omega_1)$ with trace $\gamma_{0,\Sigma_0}^{-}u_1 = pg$ and

$$\|pg\|_{H^{1/2}(\Sigma_0)} \leq c\|u_1\|_{H^1(\Omega_1)}. \quad (3)$$

Analogously there exists the function $u_2 \in H^1(\Omega_2)$ that $\gamma_{0,\Sigma_0}^{+}u_2 = pg$. Thus we have function $u \in H^1(\Omega_+)$ where u_i are the restrictions of u to Ω_i, $i = 1, 2$.

Then from (3) we obtain

$$\|g\|_{H^{1/2}(S)} = \inf_{pg\in H^{1/2}(\Sigma_0)} \|pg\|_{H^{1/2}(\Sigma_0)} \leq c\|u_1\|_{H^1(\Omega_1)} \leq c\|u\|_{H^1(\Omega_+)}.$$

Here c - some positive constant. □

3. Boundary value problem with transmision boundary condition and it's variational formulation

Let us state the following boundary value problem in domain Ω.

Problem T. Find a function $u \in H^1(\Omega, L)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega,$$

$$[\gamma_{0,S}]u = 0, \quad [\gamma_{1,S}]u + \lambda\gamma_{0,S}^- u = f,$$

$$\gamma_{0,\Sigma}^+ u = g.$$

Here $f \in H_{00}^{-1/2}(S)$, $g \in H^{1/2}(\Sigma)$ and $\lambda \in C(\bar{S})$ are given.

A partial case of the problem T when $\gamma_{0,\Sigma}^+ u = 0$ we denote as problem T_0.

We can connect with problem T_0 the next variational problem.

Problem VT_0. Find a function $u \in H_0^1(\Omega_+) = \{u \in H^1(\Omega_+) : \gamma_{0,\Sigma}^+ u = 0\}$ that satisfies

$$b(u, v) = l(v)$$

for every $v \in H_0^1(\Omega_+)$.

Here

$$b(u, v) = (\nabla u, \nabla v)_{L_2(\Omega_+)} + (\lambda\gamma_{0,S}^- u, \gamma_{0,S}^- v)_{L_2(S)},$$

$$l(v) = \langle f, \gamma_{0,S}^- v\rangle. \tag{4}$$

Lemma 2. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then bilinear form $b(u, v) : H_0^1(\Omega_+)\times H_0^1(\Omega_+) \to \mathbb{R}$ is continuous and $H_0^1(\Omega_+)$-elliptic.*

Proof. Since trace map $\gamma_{0,S}^- : H_0^1(\Omega_+) \to H^{1/2}(S)$ is continuous we have

$$|(\lambda\gamma_{0,S}^- u, \gamma_{0,S}^- v)_{L_2(S)}| \leq M\|\gamma_{0,S}^- u\|_{L_2(S)}\|\gamma_{0,S}^- v\|_{L_2(S)} \leq$$

$$\leq M\|\gamma_{0,S}^- u\|_{H^{1/2}(S)}\|\gamma_{0,S}^- v\|_{H^{1/2}(S)} \leq Mc\|u\|_{H_0^1(\Omega_+)}\|v\|_{H_0^1(\Omega_+)},$$

where $M = \max_{x\in\bar{S}} |\lambda(x)|$.

$$|(\nabla u, \nabla v)_{L_2(\Omega_+)}| \leq \|\nabla u\|_{L_2(\Omega_+)}\|\nabla v\|_{L_2(\Omega_+)} \leq \|u\|_{H_0^1(\Omega_+)}\|v\|_{H_0^1(\Omega_+)}.$$

Thus we obtain

$$|b(u, v)| \leq (Mc + 1)\|u\|_{H_0^1(\Omega_+)}\|v\|_{H_0^1(\Omega_+)}.$$

If $\lambda(x) \geq 0$, $x \in \bar{S}$, then using Friedreich's inequality in $H_0^1(\Omega_+)$ we can get

$$b(u, u) = \|u\|^2_{L_2(\Omega_+)} + \|\lambda^{1/2}\gamma_{0,S}^- u\|^2_{L_2(S)} \geq c\|u\|^2_{H_0^1(\Omega_+)}.$$

Thus $b(u, v)$ is $H_0^1(\Omega_+)$ - elliptic. Here c - some positive constants which don't depend on u and v. □

Theorem 1. *Problems T_0 and VT_0 are equivalent.*

Proof. Let u be a solution of the problem T_0. It means that $u \in H^1(\Omega, L)$ and $[\gamma_{0,S}]u = 0$, $\gamma^+_{0,\Sigma}u = 0$. Thus $u \in H^1_0(\Omega_+)$. From the first Green's formula (2) we have $b(u, v) = l(v)$ for every $v \in H^1_0(\Omega_+)$. Thus u is a solution of the problem VT_0.

Let now $u \in H^1_0(\Omega_+)$ be a solution of the problem VT_0. Then for every $v \in H^1_0(\Omega_+)$ we have

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = \langle f - \lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v\rangle. \tag{5}$$

By definition $\langle Lu, v\rangle = (\nabla u, \nabla v)_{L_2(\Omega_+)}$ for every $u \in H^1(\Omega_+)$ and $v \in H^1_0(\Omega_+)$. Here $Lu \in H^{-1}(\Omega_+) = (H^1_0(\Omega_+))'$. If $v \in C^\infty_0(\Omega)$ from (5) we can get the following relation:

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = \langle Lu, v\rangle = 0.$$

It means that $Lu \in H^{-1}(\Omega) = (H^1_0(\Omega))'$ and $Lu = 0$ in Ω.

Since $u \in H^1_0(\Omega_+)$ it follows that $[\gamma_{0,S}]u = 0$. Then from the first Green's formula (2) for arbitrary $v \in H^1_0(\Omega_+)$ we can get:

$$\langle[\gamma_{1,S}]u - f + \lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v\rangle = 0.$$

The trace map $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is surjective. Thus $\langle[\gamma_{1,S}]u - f + \lambda\gamma^-_{0,S}u, g\rangle = 0$ for arbitrary $g \in H^{1/2}(S)$. It gives us that $[\gamma_{1,S}]u + \lambda\gamma^-_{0,S}u = f$ and as a consequence we obtain that function u is a solution of the problem T_0. □

Theorem 2. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then problem VT_0 has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$.*

Proof. Lemma 2 gives us that the bilinear form $b(u, v) : H^1_0(\Omega_+)\times H^1_0(\Omega_+) \to \mathbb{R}$ is continuous and $H^1_0(\Omega_+)$-elliptic

It's easy to show that the functional $l : H^1_0(\Omega_+) \to \mathbb{R}$ given by (4) is continuous. Since the trace map $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is continuous we have:

$$|l(v)| = |\langle f, \gamma^-_{0,S}v\rangle| \leq \|f\|_{H^{-1/2}_{00}(S)}\|\gamma^-_{0,S}v\|_{H^{1/2}(S)} \leq c\|f\|_{H^{-1/2}_{00}(S)}\|v\|_{H^1_0(\Omega_+)},$$

where c - some positive constant which does not depend on v. Then by the Lax-Milgram Lemma we obtain what was to be proved. □

Theorem 3. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then problem T has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$ and $g \in H^{1/2}(\Sigma)$.*

Proof. Let function $w \in H^1(\Omega_+)$ satisfies $Lw = 0$ in Ω_+ and $\gamma^+_{0,\Sigma}w = g$. Then $[\gamma_{0,S}]w = 0$ and $[\gamma_{1,S}]w = 0$. As a corollary of theorem 1 and theorem 2 we obtain that the problem T_0 has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$ if $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$. It means that there exists a solution u_0 of the problem T_0 with boundary condition $[\gamma_{1,S}]u_0 + \lambda\gamma^-_{0,S}u_0 = f - \lambda\gamma^-_{0,S}w$. Then it's easy to verify that the function $u = u_0 - w \in H^1(\Omega)$ is a solution of the problem T. □

Let us note that our approach remains true when $S = \bigcup_{i=1}^n S_i$, where S_i are open Lipshitz curves without common points.

4. System of boundary integral equations

Let $Q(x,y)=\frac{1}{2\pi}\ln\frac{1}{|x-y|}$ - be fundamental solution of the operator $L=-\Delta$. Then the solution u of the problem T with condition $\gamma^{-}_{0,\Sigma}u=\gamma^{+}_{0,\Sigma}u$ has the following integral representation

$$u(x)=V\tau(x)+V_{\Sigma}\mu(x),\qquad x\in\Omega_{+},$$

where $\tau=[\gamma_{1,S}]u$, $\mu=[\gamma_{1,\Sigma}]u$,

$$V\tau(x)=\int\limits_{S}Q(x,y)\tau(y)ds_y,\quad V_{\Sigma}\mu(x)=\int\limits_{\Sigma}Q(x,y)\mu(y)dy.$$

Using boundary conditions we can reduce problem T to the following system of boundary integral equations:

$$\begin{cases}\tau+\lambda K\tau+\lambda\gamma^{+}_{0,S}V_{\Sigma}\mu=f,\\ \gamma^{+}_{0,\Sigma}V\tau+K_{\Sigma}\mu=g,\end{cases}\tag{6}$$

where

$$K\tau(x)=\int\limits_{S}Q(x,y)\tau(y)dS_y,\quad \gamma^{+}_{0,S}V_{\Sigma}\mu(x)=\int\limits_{\Sigma}Q(x,y)\mu(y)dS_y,\quad x\in S,$$

$$K_{\Sigma}\mu(x)=\int\limits_{\Sigma}Q(x,y)\mu(y)dS_y,\quad \gamma^{+}_{0,\Sigma}V\tau(x)=\int\limits_{S}Q(x,y)\tau(y)dS_y,\quad x\in\Sigma.$$

We use collocation method for solving of obtained system (6). Let us denote by N_S and N_Σ number of boundary elements of the second order given upon curves S and Σ respectively. Finally we derive the following system of linear algebraic equations:

$$\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}\widetilde{\tau}\\ \widetilde{\mu}\end{pmatrix}=\begin{pmatrix}\widetilde{f}\\ \widetilde{g}\end{pmatrix}.$$

Here

$$A_{11}=\left\{\delta_{ij}+\lambda(x_i)\int\limits_{S_j}Q(x_i,y)ds_y\right\},\ i,j=\overline{1,N_S},$$

$$A_{12}=\left\{\lambda(x_i)\int\limits_{\Sigma_j}Q(x_i,y)ds_y\right\},\ i=\overline{1,N_S},\ j=\overline{1,N_\Sigma},$$

$$A_{21}=\left\{\int\limits_{S_j}Q(x_i,y)ds_y\right\},\ i=\overline{1,N_\Sigma},\ j=\overline{1,N_S},$$

$$A_{22}=\left\{\int\limits_{\Sigma_j}Q(x_i,y)ds_y\right\},\ i,j=\overline{1,N_\Sigma},$$

$$\widetilde{\tau} = (\tau_1, \cdots, \tau_{N_S}), \qquad \widetilde{\mu} = (\mu_1, \cdots, \mu_{N_\Sigma}),$$
$$\widetilde{f} = (f(x_1), \cdots, f(x_{N_S})), \qquad \widetilde{g} = (g(x_1), \cdots, g(x_{N_\Sigma})),$$

x_i – collocation points on S or Σ.

Approximate meaning of searching solution of the problem T we can get from the next expression:

$$u(x) = \sum_{i=1}^{N_S} \tau_i \int\limits_{S_i} Q(x,y)ds_y + \sum_{i=1}^{N_\Sigma} \mu_i \int\limits_{\Sigma_i} Q(x,y)ds_y.$$

5. Numerical examples

Example 1. We consider the domain Ω bounded by circle Σ of the radius $R = 2$ and with open curve $S = \{(x_1, x_2) : x_2 = x_1, -1 < x_1 < 1\}$ (see Fig. 1):

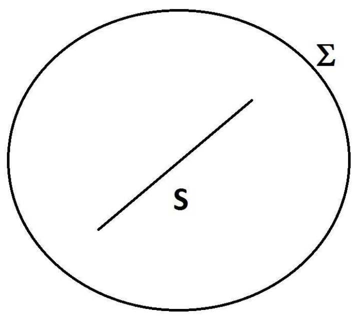


FIG. 1

The obtained numerical result for given meaning of λ, f and g is presented in Fig. 2a and Fig. 2b.

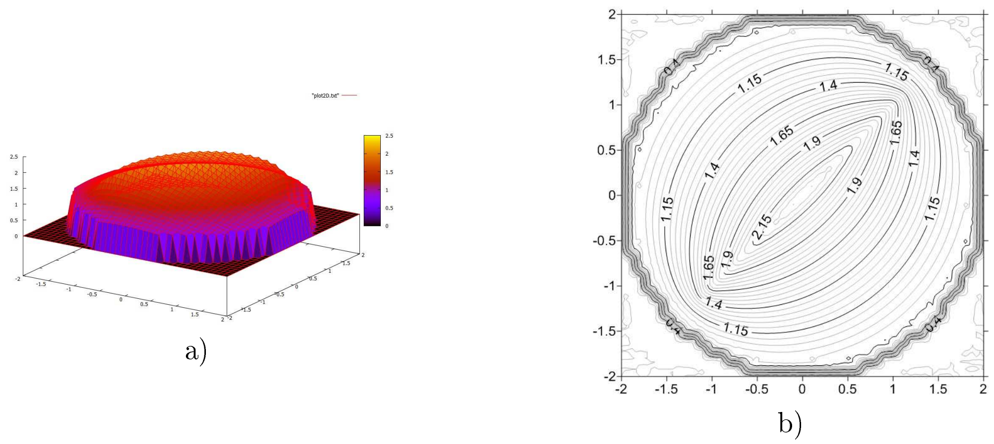
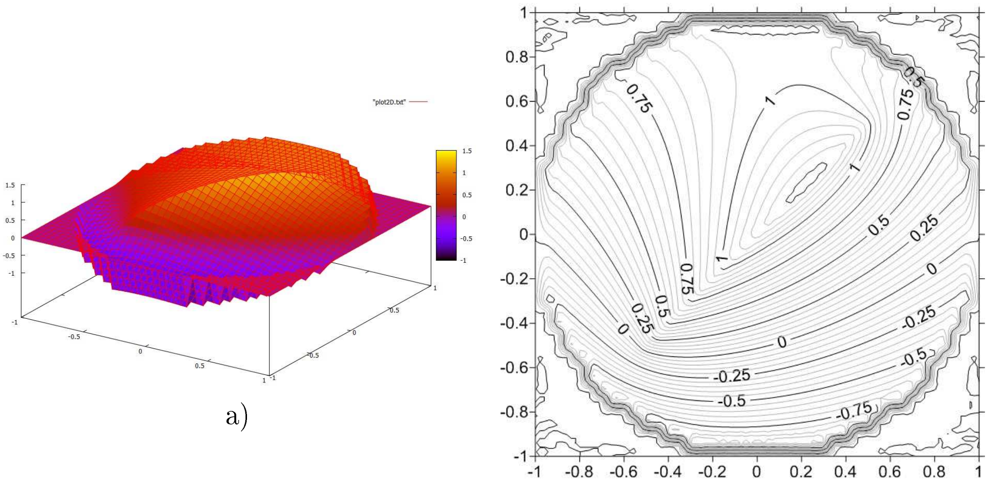


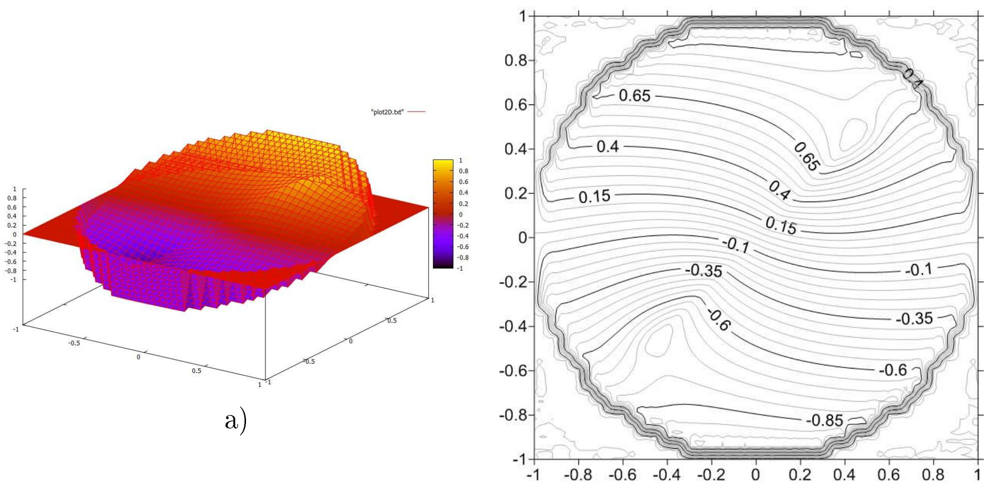
FIG. 2. $\lambda = 1$, $g = 1$, $f = 5$, $N_\Sigma = 800$, $N_S = 160$

If we take another meanings of functions g and f we can get the following results (see Fig. 3a, Fig. 3b, Fig. 4a and Fig. 4b):



a) b)

FIG. 3. $\lambda = 1$, $g = x_2$, $f = 5$, $N_\Sigma = 800$, $N_S = 160$



a) b)

FIG. 4. $\lambda = 1$, $g = x_2$, $f = 10x_2$, $N_\Sigma = 800$, $N_S = 160$

Example 2. We consider the next domain where S consists of two parts as it presented on Fig. 5

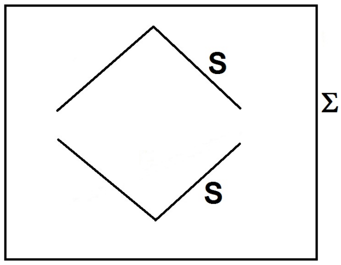
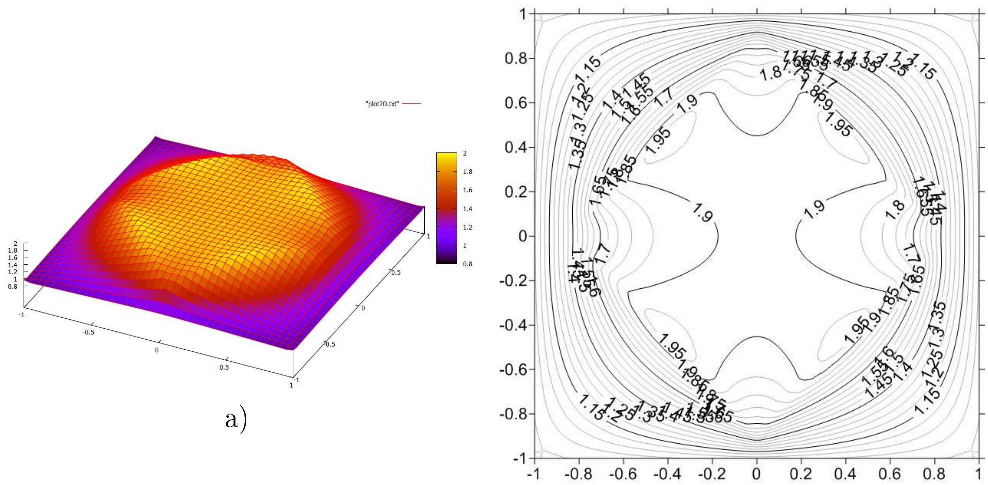


FIG. 5

Numerical result for given meanings of λ, f and g for this example is presented in Fig. 6a and Fig. 6b.



a) b)

FIG. 6. $\lambda = 1$, $g = 1$, $f = 1$, $N_\Sigma = 640$, $N_S = 320$

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EXPONENTIALLY CONVERGENT METHOD FOR DIFFERENTIAL EQUATION IN BANACH SPACE WITH A BOUNDED OPERATOR IN NONLOCAL CONDITION

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Резюме. Розглядається двоточкова нелокальна задача для диференціального рівняння першого порядку з необмеженим операторним коефіцієнтом в банаховому просторі X. В нелокальній умові міститься обмежений операторний коефіцієнт. Побудовано та обгрунтовано новий експоненціально збіжний метод у випадку, коли операторний коефіцієнт A у рівнянні є секторіальним і виконанні умови існування та єдиності розв'язку. Цей метод грунтується на зображенні операторних функцій за допомогою інтеграла Данфорда-Коші вздовж гіперболи, що охоплює спектр оператора A, та відповідній квадратурній формулі, що містить невелику кількість резольвент. Ефективність запропонованого методу демонструється за допомогою чисельних розрахунків.

Abstract. The two-pointed nonlocal problem for the first order differential equation with an unbounded operator coefficient in a Banach space X is considered. The nonlocal condition involves a bounded operator coefficient. A new exponentially convergent method is proposed and justified in the case when the operator coefficient A in equatuion is strongly positive and some existence and uniqueness conditions are fulfilled. This method is based on representations of operator functions by a Dunford-Cauchy integral along a hyperbola enveloping the spectrum of A and on the proper quadratures involving short sums of resolvents. The efficiency of proposed method is demonstrated by numerical examples.

1. Introduction

Problems with nonlocal conditions arise in many applications particulary in the theory of physics of plasma [12], nuclear physics [9], waveguides [7] etc. The nonlocal problems for a differential equation with various nonlocal conditions are also interesting from theoretical point of view and are ones of the important topics in the study of differential equations.

Differential equations with operator coefficients in some Hilbert or Banach space can be considered as meta-models for systems of partial or ordinary differential equations and are suitable for investigations using tools of the functional analysis (see e.g. [8, 11]). Nonlocal problems often are considered within this framework [1–3, 18, 19].

Key words. Nonlocal problem; differential equation with an operator coefficient in Banach space; operator exponential; exponentially convergent methods.

In this work we consider the following nonlocal two-pointed problem:

$$\begin{aligned} &u'_t + Au = f(t), \quad t \in [0, T] \\ &u(0) + Bu(T) = u_0, \quad 0 < T, \end{aligned} \tag{1}$$

where $B : X \to X$ is a bounded operator, $f(t)$ is a given vector-valued function with values in Banach space X, $u_0 \in X$. The operator A with domain $D(A)$ in Banach space X is assumed to be a densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin and with a resolvent that decays inversely proportional to $|z|$ at the infinity (see estimate (2) below).

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [5, 10, 13, 14, 16, 17] and the references therein). Methods from [5, 10, 14, 16, 17] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type $\mathcal{O}(\mathrm{e}^{-N^\alpha})$, $\alpha > 0$ with respect to a discretization parameter $N \to \infty$. For a given tolerance ε such methods provide optimal or nearly optimal computational complexity [4]. One of the possible ways to obtain exponentially convergent approximations to abstract differential equations is based on a representation of the solution through the Dunford-Cauchy integral along a parametrized path enveloping the spectrum of the operator coefficient and choosing a proper quadrature for this integral. In such way we obtain a short sum of resolvents. Since the treatment of such resolvents is usually the most time consuming part of any approximation this leads to a low-cost naturally parallelization techniques. Parameters of the algorithms from [5, 10, 14] were optimized in [20, 21] to improve the convergence rate.

Exponentially convergent method was constructed recently for nonlocal m-point problem for the first order differential equation with an unbounded coefficient in Banach space in [3]. But unlike this work there were considered the case of scalar coefficients in nonlocal condition. The aim of this paper is to construct an exponentially convergent approximation to the problem for a differential equation with two-pointed nonlocal condition with a bounded operator in abstract setting (1). The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical method for the homogeneous problem (1) is proposed in section 3. The main result of this section is theorem 1 about the exponential convergence rate of the proposed discretization.

2. Existence and representation of the solution

Let the operator A in (1) be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector. Additionally outside the sector and on its boundary Γ_Σ the following estimate for the resolvent holds true

$$\left\|(zI - A)^{-1}\right\| \le \frac{M}{1 + |z|}. \tag{2}$$

Let us assume that operator B is bounded in Banach space X, i.e. $\|B\| \leq c < \infty$.

The hyperbola

$$\Gamma_0 = \{z(\xi) = \rho_0 \cosh\xi - ib_0 \sinh\xi : \ \xi \in (-\infty, \infty), \ b_0 = \rho_0 \tan\varphi\} \qquad (3)$$

is called a spectral hyperbola. It has a vertex at $(\rho_0, 0)$ and asymptotes which are parallel to the rays of the spectral angle Σ. The numbers ρ_0, φ are called the spectral characteristics of A.

A convenient representation of operator functions is the one through the Dunford-Cauchy integral (see e.g. [8, 11]) where the integration path plays an important role. We choose the following hyperbola

$$\Gamma_I = \{z(\xi) = a_I \cosh\xi - ib_I \sinh\xi : \ \xi \in (-\infty, \infty)\}, \qquad (4)$$

as the integration contour which envelopes the spectrum of A.

One can reduce problem (1) to homogeneous using the following way. Let $u = v + w$, where v is a solution to the problem

$$\begin{aligned} &v'_t + Au = f(t), \quad t \in [0, T] \\ &u(0) = 0. \end{aligned}$$

Namely it has a representation

$$v(t) = \int_0^t \mathrm{e}^{-A(t-\tau)} f(\tau) d\tau. \qquad (5)$$

Then for $w(t)$ we obtain the problem

$$\begin{aligned} &w'_t + Aw = 0, \quad t \in [0, T] \\ &w(0) + Bw(T) = u_0 - B\int_0^T \mathrm{e}^{-A(T-\tau)} f(\tau) d\tau = \tilde{u}_0, \quad 0 < T. \end{aligned}$$

Note that exponentially convergent method for approximating $v(t)$ from (5) was developed in [6] (see also [4]). So, we can consider homogeneous problem (1) ($f(t) \equiv 0$).

According to the Hille-Yosida-Phillips theorem [22] the strongly positive operator A generates a one parameter semigroup $T(t) = \mathrm{e}^{-tA}$ and solution to (1) (homogeneous case) can be represented by

$$u(t) = \mathrm{e}^{-At} u(0). \qquad (6)$$

Combining the nonlocal condition from (1) and (6) we obtain

$$u(0) + B\mathrm{e}^{-AT} u(0) = u_0, \qquad (7)$$

from where we have

$$u(0) = \left[I + B\mathrm{e}^{-AT}\right]^{-1} u_0,$$

in the case when $\left[I + B\mathrm{e}^{-AT}\right]^{-1}$ exists. Here I is an identity operator. So, using (6) we obtain

$$u(t) = \mathrm{e}^{-At} \left[I + B\mathrm{e}^{-AT}\right]^{-1} u_0. \qquad (8)$$

Let us looking for existing conditions for $\left[I + \mathrm{e}^{-AT}B\right]^{-1}$. We have

$$\left\| \left[I + \mathrm{e}^{-AT}B\right]^{-1} \right\| \le \left(1 - \|\mathrm{e}^{-AT}B\|\right)^{-1} \le (1 - \|B\|)^{-1} \le c < \infty,$$

in the case

$$\|B\| < 1. \tag{9}$$

Remark 5. *It is possible to obtain weaker conditions than* (9) *in the case when the operator* A *is positive definite and selfadjoint* $A = A^* \ge \lambda_0 I$, $\lambda_0 > 0$. *For example if* $B = A$ *then we have using spectral integral representation*

$$\|B\mathrm{e}^{-AT}\| = \left\| \int_{\lambda_0}^{\infty} \mathrm{e}^{-\lambda T} \lambda dE_\lambda \right\| \le \frac{\mathrm{e}^{-1}}{T} \int_{\lambda_0}^{\infty} \|dE_\lambda\| = \frac{\mathrm{e}^{-1}}{T}.$$

Therefore, for $T > \mathrm{e}^{-1}$ *we have*

$$\left\| \left[I + \mathrm{e}^{-AT}B\right]^{-1} \right\| \le \left[1 - \|\mathrm{e}^{-AT}A\|\right]^{-1} < \left[1 - \frac{\mathrm{e}^{-1}}{T}\right]^{-1} = \frac{T}{T - \mathrm{e}^{-1}} < \infty.$$

3. Numerical approximation

Our aim in this section is to construct an exponentially convergent method for the solution to homogeneous problem (1) with assumption (9). Additionally we assume that the operators A and B are commutative: $AB = BA$.

Using the Dunford-Cauchy representation of $u(t)$ (see [11]) analogously to [4] we obtain

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} \mathrm{e}^{-zt} \left[I + \mathrm{e}^{-zT}B\right]^{-1} (zI - A)^{-1} u_0 dz \tag{10}$$

Representation (10) makes sense only when the function $\mathrm{e}^{-zt}\left[I + \mathrm{e}^{-zT}B\right]^{-1}$ is analytic in the region enveloped by Γ_I. Let us show, that condition (9) guaranty this analyticity [8].

Actually, the analyticity of $\mathrm{e}^{-zt}\left[I + \mathrm{e}^{-zT}B\right]^{-1}$ might only be violated when $\mathrm{e}^{-zT}B = -I$, since in this case the function becomes unbounded. It is easy to see that for an arbitrary z we have

$$\left\| I + Be^{-zT} \right\| \ge |1 - \|B\|| > 0,$$

provided that (9) holds true.

We modify the representation of $u(t)$ to obtain numerical stability for small t as follows (see [4]):

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} \mathrm{e}^{-zt} \left[I + \mathrm{e}^{-zT}B\right]^{-1} \left[(zI - A)^{-1} - \frac{1}{z} I\right] u_0 dz. \tag{11}$$

After discretization of the integral such modified resolvent provides better convergence speed than (10) in a neighborhood of $t = 0$ (see [4, 6] for details).

Parameterizing the integral (11) by (4) we get

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t, \xi) d\xi, \tag{12}$$

with

$$\mathcal{F}(t,\xi) = F_A(t,\xi)u_0,$$

$$F_A(t,\xi) = \mathrm{e}^{-z(\xi)t} z'(\xi)[I + B\mathrm{e}^{-z(\xi)T}]^{-1}\left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I\right],$$

$$z'(\xi) = a_I \sinh\xi - ib_I \cosh\xi.$$

Supposing $u_0 \in D(A^\alpha),\ 0 < \alpha < 1$ it was shown in [4,6] that

$$\|\mathrm{e}^{-z(\xi)t} z'(\xi)\left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I\right]u_0\|$$

$$\leq (1+M)K\frac{b_I}{a_I}\left(\frac{2}{a_I}\right)^\alpha \mathrm{e}^{-a_I t\cosh\xi - \alpha|\xi|}\|A^\alpha u_0\|,\ \xi \in \mathbb{R},\ t \geq 0.$$

The part responsible for the nonlocal condition in (12), can be estimated in the following way

$$\left\|\left(I + B\mathrm{e}^{-z(\xi)T}\right)^{-1}\right\| \leq (1 - \|B\|)^{-1} = Q.$$

Thus, we obtain the following estimate for $\mathcal{F}(t,\xi)$ using commutative property of operators A and B:

$$\|\mathcal{F}(t,\xi)\| \leq Q(1+M)K\frac{b_I}{a_I}\left(\frac{2}{a_I}\right)^\alpha \mathrm{e}^{-a_I t\cosh\xi - \alpha|\xi|}\|A^\alpha u_0\|,\ \xi \in \mathbb{R},\ t \geq 0. \quad (13)$$

Further, we have to estimate a strip around the real axis where the function $\mathcal{F}(t,\xi)$ permit analytical extension (with respect to ξ). The analyticity of function $\mathcal{F}(t,\xi + i\nu)$, in the strip

$$D_{d_1} = \{(\xi,\nu) : \xi \in (-\infty,\infty), |\nu| < d_1/2\},$$

with some d_1 could be violated if the resolvent or the part related to the nonlocal condition become unbounded. To avoid this we have to choose d_1 so that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola set $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola is going through the origin $(0,0)$. For $\nu = d_1/2$ it coincides with the spectral hyperbola and therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector.

The above requirements are fulfilled when (see [4])

$$d_1 = \arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}}\right) - \varphi, \quad (14)$$

where $\cos\varphi = \frac{\rho_0}{\sqrt{\rho_0^2+b_0^2}}$, $\sin\varphi = \frac{b_0}{\sqrt{\rho_0^2+b_0^2}}$. And for a_I, b_I

$$\begin{aligned} a_I &= \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi\right) \\ &= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi}, \\ b_I &= \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi\right) \\ &= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi}. \end{aligned} \tag{15}$$

For a_I and b_I defined as above the vector valued function $\mathcal{F}(t,w)$ is analytic in the strip D_{d_1} with respect to $w = \xi + i\nu$ for any $t \geq 0$.

Similarly to [15] (see [4]), we introduce the space $\mathbf{H}^p(D_d)$, $1 \leq p \leq \infty$ of all vector-valued functions $\mathcal{F}$ analytic in the strip

$$D_d = \{z \in \mathbb{C} : -\infty < \Re z < \infty, |\Im z| < d\},$$

equipped by the norm

$$\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} = \begin{cases} \lim_{\epsilon\to 0}(\int_{\partial D_d(\epsilon)}\|\mathcal{F}(z)\|^p|dz|)^{1/p} & \text{if } 1 \leq p < \infty, \\ \lim_{\epsilon\to 0}\sup_{z\in\partial D_d(\epsilon)}\|\mathcal{F}(z)\| & \text{if } p = \infty, \end{cases}$$

where

$$D_d(\epsilon) = \{z \in \mathbb{C} : |\mathrm{Re}(z)| < 1/\epsilon, |\mathrm{Im}(z)| < d(1-\epsilon)\}$$

and $\partial D_d(\epsilon)$ is the boundary of $D_d(\epsilon)$.

Similarly to [4] we have estimate for $\|\mathcal{F}(t,w)\|$

$$\begin{aligned} &\|\mathcal{F}(t,\cdot)\|_{\mathbf{H}^1(D_{d_1})} \leq \|A^\alpha u_0\|[C_-(\varphi,\alpha) \\ &+ C_+(\varphi,\alpha)]\int_{-\infty}^{\infty} e^{-\alpha|\xi|}d\xi = C(\varphi,\alpha)\|A^\alpha u_0\| \end{aligned} \tag{16}$$

with

$$\begin{aligned} C(\varphi,\alpha) &= \frac{2}{\alpha}[C_+(\varphi,\alpha) + C_-(\varphi,\alpha)], \\ C_\pm(\varphi,\alpha) &= (1+M)QK\tan\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)\left(\frac{2\cos\varphi}{\rho_0\cos\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)}\right)^\alpha. \end{aligned}$$

Note that the influence of both the smoothness parameter of u_0 given by α and of the spectral characteristics of the operator A given by φ and ρ_0 is accounted by that fact, that the constant $C(\varphi,\alpha)$ from (15) tends to ∞ if $\alpha \to 0$, $\varphi \to \pi/2$ or $\rho_1 \to 0$ (in this case due to (14) $d_1 \to \frac{\pi}{2} - \varphi$).

We approximate integral (12) by the following Sinc-quadrature [4, 6, 15]:

$$u_N(t) = \frac{h}{2\pi i}\sum_{k=-N}^{N}\mathcal{F}(t, z(kh)), \tag{17}$$

with an error

$$\|\eta_N(\mathcal{F},h)\| = \|u(t) - u_{h,N}(t)\|$$

$$\le \|u(t) - \frac{h}{2\pi i}\sum_{k=-\infty}^{\infty}\mathcal{F}(t,z(kh))\| + \|\frac{h}{2\pi i}\sum_{|k|>N}\mathcal{F}(t,z(kh))\|$$

$$\le \frac{1}{2\pi}\frac{e^{-\pi d_1/h}}{2\sinh(\pi d_1/h)}\|\mathcal{F}\|_{\mathbf{H}^1(D_{d_1})}$$

$$+\frac{C(\varphi,\alpha)h\|A^\alpha u_0\|}{2\pi}\sum_{k=N+1}^{\infty}\exp[-a_I t\cosh(kh) - \alpha kh]$$

$$\le \frac{c\|A^\alpha u_0\|}{\alpha}\left\{\frac{e^{-\pi d_1/h}}{\sinh(\pi d_1/h)} + \exp[-a_I t\cosh((N+1)h) - \alpha(N+1)h]\right\},$$

where the constant c does not depend on h, N, t. Equalizing the both exponentials for $t = 0$ implies

$$\frac{2\pi d_1}{h} = \alpha(N+1)h,$$

or after the transformation

$$h = \sqrt{\frac{2\pi d_1}{\alpha(N+1)}}. \tag{18}$$

With this step-size the following error estimate holds true

$$\|\eta_N(\mathcal{F},h)\| \le \frac{c}{\alpha}\exp\left(-\sqrt{\frac{\pi d_1 \alpha}{2}(N+1)}\right)\|A^\alpha u_0\|, \tag{19}$$

where the constant c independent of t, N. In the case $t > 0$ the first summand in the argument of $\exp[-a_I t\cosh((N+1)h) - \alpha(N+1)h]$ from the estimate for $\|\eta_N(\mathcal{F},h)\|$ contributes mainly to the error order. Setting in this case $h = c_1 \ln N/N$ with some positive constant c_1 we remain, asymptotically for a fixed t, with an error

$$\|\eta_N(\mathcal{F},h)\| \le c\left[e^{-\pi d_1 N/(c_1 \ln N)} + e^{-c_1 a_I t N/2 - c_1 \alpha \ln N}\right]\|A^\alpha u_0\|, \tag{20}$$

where c is a positive constant. Thus, we have proven the following result.

Theorem 1. *Let A be a densely defined strongly positive operator and $u_0 \in D(A^\alpha)$, $\alpha \in (0,1)$, then the Sinc-quadrature (17) represents an approximate solution of the homogeneous nonlocal value problem (1) (i.e. the case when $f(t) \equiv 0$) and possesses an exponential convergence rate which is uniform with respect to $t \ge 0$ and is of the order $\mathcal{O}(e^{-c\sqrt{N}})$ uniformly in $t \ge 0$ for $h = \mathcal{O}(1/\sqrt{N})$ (estimate (19)) and of the order $\mathcal{O}\left(max\left\{e^{-\pi d N/(c_1 \ln N)},\ e^{-c_1 a_I t N/2 - c_1\alpha \ln N}\right\}\right)$ for each fixed $t > 0$ when $h = c_1 \ln N/N$ (estimate (20)).*

TABL. 1. The error for $x = 0.5,\ t = 0.5$.

N	$\varepsilon_{1,N}$	$\varepsilon_{2,N}$
8	0.4686576088595737062e-1	0.1900886270925846e-2
16	0.934021577137014178e-2	0.852946984325721275711e-4
32	0.1546349721567053042e-3	0.810358320985172283872e-5
64	0.0159641801061596051e-3	0.01035505780238307696e-5
128	0.735484912605954949e-5	0.91841759148488051333e-6
256	0.146908016254907436e-7	0.24806555113840622551e-7
512	0.8577765610e-8	0.1165963141e-8
1024	0.7339799837e-11	0.1591565422e-11

TABL. 2. The estimate of c

N	c
4	2.37265251538874558858 7496
8	1.12014873279544951562 7946
16	1.45874197676515316544 5005
32	1.52764892460113013125 0452
64	1.47679459638759175903 2900
128	1.49993501137307573607 5927
256	1.50659733908160984471 7370

4. NUMERICAL EXAMPLE

We consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) + Bu(x,1) = u_0,$$

with

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \tag{21}$$

$$u_0(x,t) = \begin{pmatrix} (1+0.2\mathrm{e}^{-\pi^2})\sin(\pi x) + 0.1\mathrm{e}^{-4\pi^2}\sin(2\pi x) \\ 0.1\mathrm{e}^{-\pi^2}\sin(\pi x) + (1+0.4\mathrm{e}^{-4\pi^2})\sin(2\pi x) \end{pmatrix} \tag{22}$$

It is easy to check that exact solution is

$$u(x,t) = \begin{pmatrix} \sin(\pi x) \\ \sin(2\pi x) \end{pmatrix}, \tag{23}$$

The error of computation is presented in Tabl. 1.

Due to Theorem 1 the error should not be greater then $\varepsilon_N = \mathcal{O}\left(\mathrm{e}^{-c\sqrt{N}}\right)$. The constant c in the exponent can be estimated using the following a-posteriori relation:

$$c = \ln\left(\frac{\varepsilon_N}{\varepsilon_{2N}}\right)(\sqrt{2}-1)^{-1}N^{-1/2} = \ln(\mu_N)(\sqrt{2}-1)^{-1}N^{-1/2}.$$

The numerical results are presented in Tabl. 2. Note that the constant can be estimated as $c \approx 1.5$ when $N \to \infty$.

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INTERPOLATION FORMULAS FOR FUNCTIONS, DEFINED ON THE SETS OF MATRICES WITH DIFFERENT MULTIPLICATION RULES

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Резюме. Розглядається задача інтерполяції функції від матриці у випадку множення за правилами Йордана, Адамара, Фробеніуса, Кронекера і Лапласа. Отримано новий клас інтерполяційних многочленів Лагранжа і Ньютона фіксованого степеня для функцій, визначених на множинах скінченних і нескінченних матриць. Вказано вигляд операторних поліномів, для яких ці формули інваріантні.

Abstract. We consider the problem of matrix functions interpolation in the case of Jordan, Hadamard, Frobenius, Kronecker and Laplace multiplication rules. We give a new class of Lagrange and Newton interpolation polynomials of fixed degree for functions, defined on the sets of finite and infinite matrices. The type of operator polynomials, for which these formulas are invariant, is indicated.

1. Introduction

Let X be a set of square or rectangular matrices of the fixed size. The operator $F : X \to Y$, where Y is a given set, is called a function of the matrix. In particular, Y may coincide with X, may be some other set of matrices, a numerical set, a function space and others.

Approximation of functions of the matrix variables is a part of a more general problem – interpolation of operators [1–4].

General form of the interpolation formulas is determined by the structure and properties of elements of the set X, on which the interpolated function $F(A)$ is given, as well as the interpolation nodes. A number of interpolation formulas on the sets of square and rectangular matrices was obtained in the works [1, 2; 5–8].

Along with the commonly accepted operation of matrix multiplication, the other matrix multiplication rules are also used and can be applied in mathematics and its applications. Such an approach is also effective at constructing of interpolation methods for functions of matrices. In this paper the interpolation formulas, using both the ordinary matrix multiplication and the matrix multiplication by Jordan, Hadamard, Frobenius and others, are obtained.

Key words. Interpolation; matrix functions; interpolation matrix polynomial; interpolation formula of Lagrange and Newton type; matrix multiplication by Jordan, Hadamard, Frobenius and Kronecker, Laplace discrete convolution.

2. Interpolation formulas with multiplication of square matrix by Jordan

Let X be a set of square matrices of the fixed size, the operator $F : X \to X$. The Jordan product $A \circ B$ of two matrices A and B from X is defined by the following rule: $A \circ B = \frac{1}{2}(AB + BA)$. It is commutative, but not associative. So, if the Jordan product contains more than two matrices, then in some cases it is required to indicate the execution order of the multiplication in the given product for uniqueness.

Let us first consider interpolation formulas of Lagrange type of the arbitrary order, which are constructed on the basis of such rules of multiplication of square matrices. Here are three variants of the formulas for constant matrices. We denote by $l_{nk}(A)$ the product

$$l_{nk}(A) = B_{k0} \circ (A - A_0) \circ B_{k1} \circ ... \circ B_{k,k-1} \circ$$

$$\circ (A - A_{k-1}) \circ B_{kk} \circ (A - A_{k+1}) \circ B_{k,k+1} \circ ... \circ B_{k,n-1} \circ (A - A_n) \circ B_{nn},$$

where A_k $(k = 0, 1, ..., n)$ are interpolation nodes, $B_{k\nu} \equiv B_{k,\nu}$ $(k, \nu = 0, 1, ..., n)$ are arbitrarily given matrices. Let the order of execution of multiplication operation in $l_{nk}(A)$ be determined in advance. We introduce the matrix polynomials of the form

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) \circ \left\{ l_{nk}^{-1}(A_k) \circ l_{nk}(A) \right\} \tag{1}$$

$$L_{n0}(A) = \sum_{k=0}^{n} \left\{ F(A_k) \circ l_{nk}^{-1}(A_k) \right\} \circ l_{nk}(A), \tag{2}$$

in which first the multiplication operation in the curly brackets is performed. Since $l_{nk}^{-1}(A_k) \circ l_{nk}(A_\nu) = \delta_{k\nu} I$ $(k, \nu = 0, 1, ..., n)$, where $\delta_{k\nu}$ is the Kronecker symbol, than for the formula (1) in the nodes A_k the interpolation conditions $L_{0n}(A_k) = F(A_k)$ are met.

These conditions are satisfied for the formula (2), if the associator

$$\left\{ F(A_\nu),\ l_{n\nu}^{-1}(A_\nu),\ l_{n\nu}(A_\nu) \right\} = 0.$$

It takes place in virtue of the equality

$$\left\{ F(A_\nu),\ l_{n\nu}^{-1}(A_\nu),\ l_{n\nu}(A_\nu) \right\} =$$

$$= \left(F(A_\nu) \circ l_{n\nu}^{-1}(A_\nu) \right) \circ l_{n\nu}(A_\nu) - F(A_\nu) \circ \left(l_{n\nu}^{-1}(A_\nu) \circ l_{n\nu}(A_\nu) \right) =$$

$$= \left(F(A_\nu) \circ l_{n\nu}^{-1}(A_\nu) \right) \circ l_{n\nu}(A_\nu) - F(A_\nu) = 0.$$

It follows that (2) is the interpolation formula.

It is easy to check that the matrix polynomial of the n-th degree

$$L_n(A) = \sum_{k=0}^{n} F(A_k) \circ l_{kk}(A), \tag{3}$$

where

$$l_{kk}(A) = \prod_{\nu=0, \nu \neq k}^{n} B_\nu \left\{ (A - A_\nu) \circ (A_k - A_\nu)^{-1} \right\} B_\nu^{-1},$$

B_ν $(\nu = 0, 1, ..., n)$ are arbitrary invertible matrices, also satisfies the conditions $L_n(A_k) = F(A_k)$ $(k = 0, 1, ..., n)$, at that the product of matrices, indicated in curly brackets, on B_ν and B_ν^{-1} can be understood as the ordinary or in the sense of Jordan. In the both cases $l_{kk}(A_\nu) = \delta_{k\nu} I$ $(k, \nu = 0, 1, ..., n)$.

The interpolation polynomials (1)–(3) are exact for the matrix polynomials

$$P_{0n}(A) = \sum_{\nu=0}^{n} D_\nu \circ \left\{ l_{n\nu}^{-1}(A_\nu) \circ l_{n\nu}(A) \right\},$$

$$P_{n0}(A) = \sum_{\nu=0}^{n} \left\{ D_\nu \circ l_{n\nu}^{-1}(A_\nu) \right\} \circ l_{n\nu}(A), \quad P_n(A) = \sum_{\nu=0}^{n} D_\nu \circ l_{\nu\nu}(A),$$

respectively, where D_ν are arbitrary square matrices. As already mentioned, the interpolation conditions for the formula (2) are satisfied, if and only if associator

$$\left\{ F(A_k), \; l_{nk}^{-1}(A_k), \; l_{nk}(A_k) \right\} = 0 \quad (k = 0, 1, ..., n).$$

This imposes additional conditions on the operator F and the interpolation nodes.

If $n = 1$, and $B_{k\nu}$ $(k, \nu = 0, 1)$ are the identity matrices, then the formula (1) with the nodes A_0 and A_1 is reduced to the equality

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \circ \left\{ (A_1 - A_0)^{-1} \circ (A - A_0) \right\}. \tag{4}$$

It is exact (invariant) for the polynomials $P_{01}(A) = D \circ \left\{ (A_1 - A_0)^{-1} \circ A \right\} + C$, where D and C are arbitrary matrices.

In the particular case, when $A_1 - A_0 = I$, the linear interpolation formula (4) takes the form

$$L_{01}(A) = F(A_0) +$$
$$+ \frac{1}{2} \left[(F(A_1) - F(A_0))(A - A_0) + (A - A_0)(F(A_1) - F(A_0)) \right]$$

and it will be invariant for the matrix polynomials $P_1(A) = DA + AD + C$, where D and C are arbitrary fixed matrices.

Here is another formula of the linear interpolation with the multiplication by Jordan:

$$L_1(A) = F(A_0) + (A - A_0) \circ B +$$
$$+ [F(A_1) - F(A_0) - (A_1 - A_0) \circ B] \circ \left\{ (A_1 - A_0)^{-1} \circ (A - A_0) \right\},$$

where B is an arbitrary given matrix. This interpolation formula is exact for polynomials of the form

$$P_1(A) = D \circ \left\{ (A_1 - A_0)^{-1} \circ A \right\} + B \circ A + C.$$

One of the quadratic interpolation formulas of the kind (3) has the form

$$L_{21}(A) = L_{01}(A) + \left\{ (A - A_1) \circ (A_2 - A_1)^{-1} \right\} \circ \left[\left\{ (A - A_0) \circ (A_2 - A_0)^{-1} \right\} \circ \right.$$
$$\left. \circ (F(A_2) - F(A_1)) - \left\{ (A - A_0) \circ (A_1 - A_0)^{-1} \right\} \circ (F(A_1) - F(A_0)) \right],$$

where, as before, at the beginning matrices in the curly brackets are found, and then in the usual order – in square brackets; $L_{01}(A)$ is the matrix polynomial of the first degree (4). For it the following equalities $L_{21}(A_i) = F(A_i)$, $(i = 0, 1, 2)$ are valid.

Example 2.1. It is not difficult to show that the interpolation polynomial

$$L_{10}(A) = F(A_0) + \left\{(A - A_0) \circ (A_1 - A_0)^{-1}\right\} \circ [F(A_1) - F(A_0)],$$

where the function $F(A) = A^2$, and the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix},$$

has the form

$$L_{10}(A) = \frac{1}{2} A \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} A - \begin{bmatrix} 6 & 8 \\ 12 & 18 \end{bmatrix}.$$

Next, we consider the formulas of the linear and quadratic interpolation on the set of square functional matrices, which are determined by the matrix Stieltjes integrals. Let $X = C(T)$ be the set of continuous on $T = [a, b]$ square matrices; $F : X \to X$, $A_0(t)$, $A_1(t)$ be interpolation nodes from X.

On the set of matrices with the Jordan multiplication, the interpolation polynomial of the first degree with respect to the nodes $A_0(t)$ and $A_1(t)$ has the form

$$\begin{aligned} \tilde{L}_{10}(A) = F(A_0) + \int_T \left\{[A(\tau) - A_0(\tau)] \circ [A_1(\tau) - A_0(\tau)]^{-1}\right\} \circ \\ \circ d_\tau F[A_0(\cdot) + \chi(\tau, \cdot)(A_1(\cdot) - A_0(\cdot))]. \end{aligned} \tag{5}$$

In the formula (5), as before, first the multiplication operation in the curly brackets is carried out. This formula is invariant with respect to the polynomials

$$P_1(A) = K_0 + \int_T \left\{A(t) \circ [A_1(t) - A_0(t)]^{-1}\right\} \circ K(t) \circ [A_1(t) - A_0(t)]\, dt,$$

where K_0, $K(t)$ are some given matrices.

Example 2.2. The interpolation matrix polynomial of the form (5) with respect to the nodes $A_0(t)$ and $A_1(t)$ for the function $F(A) = \int_a^b A^2(t)\, dt$ takes the form

$$\tilde{L}_{10}(A) = F(A_0) + \int_a^b G[A(\tau), A_0(\tau), A_1(\tau)]\, d\tau,$$

where

$$\begin{aligned} G[A, A_0, A_1] &= \\ &= \frac{1}{4}\left\{(A - A_0)(A_1 - A_0)^{-1} + (A_1 - A_0)^{-1}(A - A_0)\right\}\left(A_1^2 - A_0^2\right) + \\ &+ \frac{1}{4}\left(A_1^2 - A_0^2\right)\left\{(A - A_0)(A_1 - A_0)^{-1} + (A_1 - A_0)^{-1}(A - A_0)\right\}. \end{aligned}$$

Next, we consider the interpolation polynomials of the arbitrary degree for functions of two matrix variables. Let $F(A, B)$ be a function of two variable square matrices A and B, the interpolation nodes $\{A_\nu, B_\nu\}$ $(\nu = 0, 1, ..., n)$ are given, where A_ν, B_ν are some square matrices. We introduce the following notations: $r_l \equiv r_l(A,B)$ is the vector with matrix coordinates $\{A - A_l, B - B_l\}$;

$$r_{l,k} \equiv r_{lk} \equiv r_{lk}(A_k, B_k) \equiv r_l(A_k, B_k)\,(l,\ k = 0,1,...,n)\,.$$

The vector r_{lk} has coordinates $\{A_k - A_l, B_k - B_l\}$. It's obvious that $r_{ll} = r_l(A_l, B_l) = 0$. Assume that

$$(r_l, r_{lk}) = (A - A_l) \circ (A_k - A_l) + (B - B_l) \circ (B_k - B_l)\,(l,\ k = 0,1,...,n)\,,$$

$$(r_{lk}, r_{lk}) = (A_k - A_l)^2 + (B_k - B_l)^2$$

and, accordingly, we denote

$$l_k(A,B) = (r_0, r_{0k}) \ldots (r_{k-1}, r_{k-1,k})\,(r_{k+1}, r_{k+1,k}) \ldots (r_n, r_{nk}) \times$$

$$\times \left[(r_{0k}, r_{0k}) \ldots (r_{k-1,k}, r_{k-1,k})\,(r_{k+1,k}, r_{k+1,k}) \ldots (r_{nk}, r_{nk})\right]^{-1}.$$

Since $l_k(A_\nu, B_\nu) = \delta_{k\nu} I$, then the matrix polynomial

$$L_{1n}(A,B) = \sum_{k=0}^{n} l_k(A,B)\,F(A_k, B_k)\,, \tag{6}$$

where the product of the matrices $l_k(A,B)$ and $F(A_k, B_k)$ on the right side of (6) may be usual or in the sense of Jordan, is also the interpolation polynomial for the function $F(A,B)$ with respect to the nodes (A_k, B_k) $(k = 0,1,...,n)$.

We give a slightly modified version of the interpolation formula of the form (6). We introduce the notations

$$\tilde{l}_k(A,B) = \prod_{\nu=0,\nu\neq k}^{n} \tilde{l}_{\nu k}(A,B)\,, \tilde{l}_{\nu k}(A,B) = (r_\nu, r_{\nu k}) \circ (r_{\nu k}, r_{\nu k})^{-1}.$$

Since $\tilde{l}_{\nu k}(A_k, B_k) = I$, $\tilde{l}_{\nu k}(A_\nu, B_\nu) = 0$, then $\tilde{l}_k(A_\nu, B_\nu) = \delta_{k\nu} I$. Thus, the formula

$$L_n(A,B) = \sum_{k=0}^{n} \tilde{l}_k(A,B)\,F(A_k, B_k)$$

is the interpolation polynomial of the degree not higher than n, for which the equalities $L_n(A_\nu, B_\nu) = F(A_\nu, B_\nu)$ $(\nu = 0,1,...,n)$ are true.

Next, we consider formulas of the other form for the linear interpolation of functions of two matrix variables on the set of constant matrix with the multiplication by Jordan. Let $F(A,B)$ be a function of matrix variables A and B; (A_i, B_i) be interpolation nodes $(i = 0,1,2)$.

We introduce the following notations:

$$\tilde{l}_0(A,B) = [(A - A_1) \circ (B_1 - B_2) - (A_1 - A_2) \circ (B - B_1)] \circ D^{-1},$$

$$\tilde{l}_1(A,B) = [(A - A_0) \circ (B_2 - B_0) - (A_2 - A_0) \circ (B - B_0)] \circ D^{-1},$$

$$\tilde{l}_2(A,B) = [(A - A_0) \circ (B_0 - B_1) - (A_0 - A_1) \circ (B - B_0)] \circ D^{-1},$$

where

$$D = (A_0 - A_1) \circ (B_1 - B_2) - (A_1 - A_2) \circ (B_0 - B_1)\,.$$

Note that the relations $\tilde{l}_i(A_j, B_j) = \delta_{ij} I$; $\tilde{l}_0(A,B) + \tilde{l}_1(A,B) + \tilde{l}_2(A,B) = I$ take place. It is not difficult to verify that for matrix polynomial of the variables A and B of the first degree of the form

$$\tilde{L}_1(A,B) = \tilde{l}_0(A,B) \circ F(A_0,B_0) + \tilde{l}_1(A,B) \circ F(A_1,B_1) + \tilde{l}_2(A,B) \circ F(A_2,B_2) \tag{7}$$

the interpolation conditions $\tilde{L}_1(A_i,B_i) = F(A_i,B_i)$ $(i = 0,1,2)$ are carried out.

3. Interpolation formulas with matrix multiplication by Hadamard

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be some matrices of the same dimension. The matrix $C = A \cdot B$ of the same size with elements $c_{ij} = a_{ij} b_{ij}$ is called the Hadamard product of the matrices A and B. It is commutative, associative and distributive with respect to the addition of matrices. The role of the identity matrix for such rule of multiplication carries the matrix J, all elements of which are equal to one. By $A^{\overset{\bullet}{-1}} = \left[\frac{1}{a_{ij}}\right]$ we denote the matrix that is inverse in the sense of Hadamard for the matrix $A = [a_{ij}]$ with the elements $a_{ij} \neq 0$.

By the definition, the n-th degree of matrix $A = [a_{ij}]$ in the sense of Hadamard, which is denoted as $A^{\overset{\bullet}{n}}$, is the matrix $A^{\overset{\bullet}{n}} = \left[a_{ij}^n\right]$, where $A^{\overset{\bullet}{n}} = J$ for $n = 0$. The function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of the matrix $A = [a_{ij}]$, analytical in a neighborhood of each element of this matrix, is defined on the set of matrices with Hadamard multiplication by the formula $f(A) = \sum_{k=0}^{\infty} a_k A^{\overset{\bullet}{k}}$ and, accordingly, it is the matrix $f(A) = [f(a_{ij})]$.

Here are the special cases of interpolation formula [8] of the form

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) \cdot l_{nk}^{\overset{\bullet}{-1}}(A_k) \cdot l_{nk}(A) =$$
$$= \sum_{k=0}^{n} \left[\frac{f_{ij}^k \left(a_{ij} - a_{ij}^0\right) \ldots \left(a_{ij} - a_{ij}^{k-1}\right)\left(a_{ij} - a_{ij}^{k+1}\right) \ldots \left(a_{ij} - a_{ij}^n\right)}{\left(a_{ij}^k - a_{ij}^0\right) \ldots \left(a_{ij}^k - a_{ij}^{k-1}\right)\left(a_{ij}^k - a_{ij}^{k+1}\right) \ldots \left(a_{ij}^k - a_{ij}^n\right)} \right], \tag{8}$$

where

$$l_{nk}(A) = (A - A_0) \cdot \ldots \cdot (A - A_{k-1}) \cdot (A - A_{k+1}) \cdot \ldots \cdot (A - A_n),$$

matrices $l_{nk}(A_k)$ do not have zero elements, matrix A and nodes A_k of the same dimension, f_{ij}^k are elements of the matrix $F(A_k)$ $(k = 0,1,\ldots,n)$. It is obvious that the equalities $L_{0n}(A_i) = F(A_i)$ $(i = 0,1,\ldots,n)$ hold.

Consider the linear case of the interpolation formula (8). Let the interpolation nodes $A_0 = \left[a_{ij}^0\right]$, $A_1 = \left[a_{ij}^1\right]$ be such that all elements of the matrix $A_0 - A_1 = \left[a_{ij}^0 - a_{ij}^1\right]$ are different from zero. Then for the formula

$$L_{01}(A) = F(A_0) \cdot (A_0 - A_1)^{\overset{\bullet}{-1}} \cdot (A - A_1) + F(A_1) \cdot (A_1 - A_0)^{\overset{\bullet}{-1}} \cdot (A - A_0)$$

or, what is the same thing, for the formula

$$L_{01}(A) = F(A_0) \cdot \left[\frac{a_{ij} - a_{ij}^1}{a_{ij}^0 - a_{ij}^1}\right] + F(A_1) \cdot \left[\frac{a_{ij} - a_{ij}^0}{a_{ij}^1 - a_{ij}^0}\right],$$

where $A = [a_{ij}]$ is current matrix variable, the interpolation conditions $L_{01}(A_i) = F(A_i)$ $(i = 0, 1)$ are fulfilled.

During the construction of interpolation formulas, based on the Hadamard multiplication of square matrices, it is useful to introduce yet another analogue of the inverse matrix. Let $A = [a_{ij}]$ be a square matrix and $a_{ii} \neq 0$. By $A^{(-1)}$ we denote the matrix, for which $A \cdot A^{(-1)} = A^{(-1)} \cdot A = I$, where I is the identity matrix in the ordinary sense of the same dimension as the matrix A. This matrix will be $A^{(-1)} = \operatorname{diag}\left[\frac{1}{a_{ii}}\right]$.

We give formulas of the linear interpolation with the ordinary and the Hadamard multiplication. Let $A = [a_{ij}]$ be some square matrix that has nonzero diagonal elements. Then for the linear interpolation formula

$$L_{01}(A) = F(A_0)\left\{(A_0 - A_1)^{(-1)} \cdot (A - A_1)\right\} + \\ + F(A_1)\left\{(A_1 - A_0)^{(-1)} \cdot (A - A_0)\right\},$$

or for the same formula in another form

$$L_{01}(A) = F(A_0)\operatorname{diag}\left[\frac{a_{ii} - a_{ii}^1}{a_{ii}^0 - a_{ii}^1}\right] + F(A_1)\operatorname{diag}\left[\frac{a_{ii} - a_{ii}^0}{a_{ii}^1 - a_{ii}^0}\right],$$

equalities $L_{10}(A_0) = F(A_0)$, $L_{10}(A_1) = F(A_1)$ hold.

We consider the case $n = 2$ of the interpolation formula (8). The quadratic interpolation formula with respect to the nodes $A_0 = \left[a_{ij}^0\right]$, $A_1 = \left[a_{ij}^1\right]$ and $A_2 = \left[a_{ij}^2\right]$, such that all elements of the matrices

$$A_0 - A_1 = \left[a_{ij}^0 - a_{ij}^1\right], A_0 - A_2 = \left[a_{ij}^0 - a_{ij}^2\right], A_1 - A_2 = \left[a_{ij}^1 - a_{ij}^2\right]$$

are different from zero, is a matrix polynomial of the form

$$L_{02}(A) = F(A_0) \cdot \left[\frac{\left(a_{ij} - a_{ij}^1\right)\left(a_{ij} - a_{ij}^2\right)}{\left(a_{ij}^0 - a_{ij}^1\right)\left(a_{ij}^0 - a_{ij}^2\right)}\right] + $$

$$+F(A_1) \cdot \left[\frac{\left(a_{ij} - a_{ij}^0\right)\left(a_{ij} - a_{ij}^2\right)}{\left(a_{ij}^1 - a_{ij}^0\right)\left(a_{ij}^1 - a_{ij}^2\right)}\right] + F(A_2) \cdot \left[\frac{\left(a_{ij} - a_{ij}^0\right)\left(a_{ij} - a_{ij}^1\right)}{\left(a_{ij}^2 - a_{ij}^0\right)\left(a_{ij}^2 - a_{ij}^1\right)}\right],$$

for which the conditions $L_{02}(A_i) = F(A_i)$ $(i = 0, 1, 2)$ are fulfilled.

Next, we give formulas of the quadratic interpolation with the ordinary and the Hadamard multiplication. Let $A = [a_{ij}]$ be some square matrix that has different from zero diagonal elements. For quadratic interpolation with the same restrictions on the nodes A_0, A_1 and A_2, as in the previous case, we have the formula

$$L_{02}(A) = F(A_0)\,\mathrm{diag}\left[\frac{\left(a_{ii}-a_{ii}^1\right)\left(a_{ii}-a_{ii}^2\right)}{\left(a_{ii}^0-a_{ii}^1\right)\left(a_{ii}^0-a_{ii}^2\right)}\right]+$$

$$+F(A_1)\,\mathrm{diag}\left[\frac{\left(a_{ii}-a_{ii}^0\right)\left(a_{ii}-a_{ii}^2\right)}{\left(a_{ii}^1-a_{ii}^0\right)\left(a_{ii}^1-a_{ii}^2\right)}\right]+F(A_2)\,\mathrm{diag}\left[\frac{\left(a_{ii}-a_{ii}^0\right)\left(a_{ii}-a_{ii}^1\right)}{\left(a_{ii}^2-a_{ii}^0\right)\left(a_{ii}^2-a_{ii}^1\right)}\right],$$

which satisfies the conditions $L_{02}(A_i) = F(A_i)$ $(i = 0, 1, 2)$.

Example 3.1. On the set of matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with matrix multiplication only in the sense of Hadamard for the function $F(A) = A^2$ with respect to the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix},$$

the interpolation polynomial

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \cdot (A_1 - A_0)^{\overset{\bullet}{-1}} \cdot (A - A_0)$$

takes the form

$$L_{01}(A) = \begin{bmatrix} 7 & 5 \\ 9 & 13 \end{bmatrix} \cdot A - \begin{bmatrix} 0 & 0 \\ 12 & 30 \end{bmatrix} = \begin{bmatrix} 7a_{11} & 5a_{12} \\ 9a_{21} & 13a_{22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 12 & 30 \end{bmatrix}.$$

For the constructed polynomial the interpolation conditions

$$L_{01}(A_0) = F(A_0) = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}, L_{01}(A_1) = F(A_1) = \begin{bmatrix} 0 & 0 \\ 6 & 9 \end{bmatrix}.$$

are also true. In the case if the interpolation nodes $A_k = \alpha_k J$, where α_k $(k = 0, 1, ..., n)$ are different in pairs numbers, then the formula (8) takes the form

$$L_n(A) =$$

$$= \sum_{k=0}^{n} \frac{(A-\alpha_0 J)\cdot ... \cdot (A-\alpha_{k-1} J)\cdot (A-\alpha_{k+1} J)\cdot ... \cdot (A-\alpha_n J)}{(\alpha_k-\alpha_0)...(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})...(\alpha_k-\alpha_n)} \cdot F(\alpha_k J).$$

Next, we consider interpolation formulas for operators, defined on the set of functional matrices. Let $X = C(T)$ be the set of continuous on $T = [a, b]$ square matrices; an operator $F : X \to X$ and $A_0(t)$, $A_1(t)$ be interpolation nodes from X. Suppose also that $A = A(t)$, interpolation nodes $A_0(t)$, $A_1(t)$ are matrices of the same order, defined on the segment $[a, b]$, and operator $F(A)$ is defined at the nodes $A_0(t)$, $A_1(t)$ and on the matrix curve $A_0(t)+\chi(\tau, t)(A_1(t) - A_0(t))$, where the function

$$\chi(\tau, t) = \begin{cases} 1, & \tau \ge t; \\ 0, & \tau < t, \end{cases} \quad \chi(a, t) \equiv 0,\ \ \chi(b, t) \equiv 1 \quad (a \le \tau, t \le b).$$

One of the linear interpolation formulas on the set of continuous on the segment $[a,b]$ matrices can be written using the Stieltjes integral in the form

$$L_{10}(A) = F(A_0) + \int_a^b [A(\tau) - A_0(\tau)] \cdot [A_1(\tau) - A_0(\tau)]^{\dot{-1}} \cdot d_\tau \times \\ \times F[A_0(t) + \chi(\tau,t)(A_1(t) - A_0(t))],$$

on condition that all elements of the matrix $A_1(t) - A_0(t)$ are different from zero on $[a,\ b]$ and in this formula integral exists. The equalities $L_{10}(A_i) = F(A_i)$ $(i = 0, 1)$ are true.

In the space $C^m[a,\ b]$ of rectangular matrices $A(t) = [a_{ij}(t)]$ of the dimension $p \times q$, for which the derivative $A^{(m)}(t) = \left[a_{ij}^{(m)}(t)\right]$ of order m is continuous on the $[a,\ b]$, we consider the matrix polynomial of the first degree

$$P_1(A) = B + \sum_{j=0}^{n} A(t_j) \cdot C_j + \sum_{k=0}^{m} \int_a^b D_k(t,s) \cdot A^{(k)}(s)ds \tag{9}$$

where $B = B(t)$, $C_j = C_j(t)$, $D_k(t,s)$ $(j = 0, 1, ...n;\ k = 0, 1, ..., m)$ are fixed $(p \times q)$-matrices.

We denote by $\sigma_{1i}(t)$ and $H_i(t)$ the matrices

$$\sigma_{1i}(t) = A_0(t) + A_1(t_i) - A_0(t_i), \quad H_i(t) = A(t) - A_0(t) - A(t_i) + A_0(t_i),$$

where t_i $(i = 0, 1, ...n)$ are given points of the segment $[a,b]$; $A_0(t)$ and $A_1(t)$ are interpolation nodes such that the matrices $A_1(t_i) - A_0(t_i)$ are reversible in the sense of Hadamard.

For the formula

$$\begin{aligned} L_1(A) &= F(A_0) + \\ &+ \frac{1}{n+1} \sum_{i=0}^{n} [A(t_i) - A_0(t_i)] \cdot [A_1(t_i) - A_0(t_i)]^{\dot{-1}} \cdot [F(\sigma_{1i}) - F(A_0)] + \\ &+ \frac{1}{n+1} \sum_{i=0}^{n} \int_0^1 \delta F[\sigma_{1i}(\cdot) + \tau(A_1(\cdot) - \sigma_{1i}(\cdot))\ ; H_i(\cdot)]d\tau \end{aligned} \tag{10}$$

the conditions $L_1(A_i) = F(A_i)$ $(i = 0,\ 1)$ hold, and it is exact for matrix polynomials of the form (9).

Really, the equation $L_1(A_0) = F(A_0)$ is satisfied, since the second and third terms in (10) become zero. Execution of interpolation condition at the second node is also easy to verify, taking into account that in this case the integral in (10) can be calculated exactly.

Let $F(A,B)$ be also a function of two matrix variables A and B; (A_i, B_i) be interpolation nodes $(i = 0, 1, 2)$. We introduce the following notations:

$$l_{10}(A,B) = [(A - A_1) \cdot (B_1 - B_2) - (A_1 - A_2) \cdot (B - B_1)] \cdot D^{\dot{-1}},$$

$$l_{11}(A,B) = [(A - A_0) \cdot (B_2 - B_0) - (A_2 - A_0) \cdot (B - B_0)] \cdot D^{\dot{-1}},$$

$$l_{12}(A,B) = [(A - A_0) \cdot (B_0 - B_1) - (A_0 - A_1) \cdot (B - B_0)] \cdot D^{\dot{-1}}.$$

Here the matrix $\overset{\bullet}{D}^{-1}$ is reversible in the sense of Hadamard for

$$D = (A_0 - A_1) \cdot (B_1 - B_2) - (A_1 - A_2) \cdot (B_0 - B_1);$$

A, B are independent variables, interpolation nodes (A_i, B_i) and values $F(A_i, B_i)$ $(i = 0, 1, 2)$ are rectangular matrices of the same dimension.

For the interpolation formula

$$\begin{aligned} L_{11}(A, B) = l_{10}(A, B) \cdot F(A_0, B_0) + \\ + l_{11}(A, B) \cdot F(A_1, B_1) + l_{12}(A, B) \cdot F(A_2, B_2) \end{aligned} \tag{11}$$

the conditions $L_{11}(A_i, B_i) = F(A_i, B_i)$ $(i = 0, 1, 2)$ are satisfied. The formula (11) is invariant with respect to matrix polynomials of the form

$$P_1(A, B) = l_{10}(A, B) \cdot C_0 + l_{11}(A, B) \cdot C_1 + l_{12}(A, B) \cdot C_2. \tag{12}$$

At that in the equation (12) arbitrary rectangular matrices C_i are of the same dimension as the matrices $F(A_i, B_i)$ $(i = 0, 1, 2)$.

Example 3.2. Let $A = [a_{ij}]$, $B = [b_{ij}]$ $(i, j = 1, 2)$ be square matrices of the second order. The interpolation formula (11) for the function $F(A, B) = (AB)^2$ with respect to the nodes

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

takes the form

$$L_{11}[A, B] = \begin{bmatrix} 8 - 8a_{11} + 4b_{11} & 1 + 4b_{12} \\ -11 + 11a_{21} + 10b_{21} & 7 + 4a_{22} - 2b_{22} \end{bmatrix}.$$

For $L_{11}[A, B]$ the interpolation conditions

$$L_{11}[A_0, B_0] = F(A_0, B_0) = \begin{bmatrix} 0 & 9 \\ 0 & 9 \end{bmatrix},$$

$$L_{11}[A_1, B_1] = F(A_1, B_1) = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix},$$

$$L_{11}[A_2, B_2] = F(A_2, B_2) = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

are true.

Note that in [1, 46 p.] the matrix Γ is constructed as a sum of the powers of the Hadamard matrices, which plays an important role in the construction of the set of interpolating polynomials in the Hilbert space and in the justification of a number of the results obtained on this set.

4. Interpolation formulas with matrix multiplication by Frobenius

Suppose that the matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ have the same dimension. Their product in the sense of Frobenius is defined as

$$A\diamondsuit B=\sum_{i,j}a_{ij}b_{ij}.$$

This operation is commutative, and its result is a scalar. Interpolation formulas for functions of matrices may be also constructed on the basis of such multiplication rule.

Let interpolation nodes A_k $(k=0,1,...,n)$ be different stationary or functional matrices, and $F(A_k)$ be given fixed matrices, which dimension may differ from the dimension of A_k, or some other mathematical objects over the field of real or complex numbers. Then in the case of rectangular matrices of the same dimension (including square matrices) for the formula

$$L_n(F;A)=\sum_{k=0}^{n}\frac{l_{nk}(A)}{l_{nk}(A_k)}F(A_k), \tag{13}$$

where

$$l_{nk}(A)=[(A-A_0)\diamondsuit(A_k-A_0)]\,...\,[(A-A_{k-1})\diamondsuit(A_k-A_{k-1})]\times$$
$$\times[(A-A_{k+1})\diamondsuit(A_k-A_{k+1})]\,...\,[(A-A_n)\diamondsuit(A_k-A_n)],$$

the equalities $L_n(F;A_\nu)=F(A_\nu)$ $(\nu=0,\ 1,\ ...,\ n)$ take place.

If the interpolation nodes A_k such that $\mathrm{tr}(A_k-A_\nu)\neq 0$ $(k,\nu=0,1,...,n)$, then on the set of square matrices for the similar formula

$$L_n(F;A)=\sum_{k=0}^{n}\frac{\tilde{l}_{nk}(A)}{\tilde{l}_{nk}(A_k)}F(A_k),$$

where

$$\tilde{l}_{nk}(A)=\mathrm{tr}(A-A_0)\,\mathrm{tr}(A_k-A_0)\cdots\mathrm{tr}(A-A_{k-1})\,\mathrm{tr}(A_k-A_{k-1})\times$$
$$\times\mathrm{tr}(A-A_{k+1})\,\mathrm{tr}(A_k-A_{k+1})\cdots\mathrm{tr}(A-A_n)\,\mathrm{tr}(A_k-A_n),$$

the same interpolation conditions are fulfilled.

Obviously, the equation (13) remains an interpolation, if $l_{nk}(A)$ is replaced by any number function $\phi_{nk}(A)$ of matrix function arguments such that $\phi_{nk}(A_k)\neq 0$ for $k=0,1,...,n$.

In particular, if $n=2$ and $n=1$, then the formula (13) takes the form

$$L_2(F;A)=\frac{[(A-A_1)\diamondsuit(A_0-A_1)]\,[(A-A_2)\diamondsuit(A_0-A_2)]}{[(A_0-A_1)\diamondsuit(A_0-A_1)]\,[(A_0-A_2)\diamondsuit(A_0-A_2)]}F(A_0)+$$
$$+\frac{[(A-A_0)\diamondsuit(A_1-A_0)]}{[(A_1-A_0)\diamondsuit(A_1-A_0)]}\frac{[(A-A_2)\diamondsuit(A_1-A_2)]}{[(A_1-A_2)\diamondsuit(A_1-A_2)]}F(A_1)+$$
$$+\frac{[(A-A_0)\diamondsuit(A_2-A_0)]\,[(A-A_1)\diamondsuit(A_2-A_1)]}{[(A_2-A_0)\diamondsuit(A_2-A_0)]\,[(A_2-A_1)\diamondsuit(A_2-A_1)]}F(A_2)$$

and

$$L_1(F;A) = \frac{(A-A_1)\diamondsuit(A_0-A_1)}{(A_0-A_1)\diamondsuit(A_0-A_1)}F(A_0) + \\ + \frac{(A-A_0)\diamondsuit(A_1-A_0)}{(A_1-A_0)\diamondsuit(A_1-A_0)}F(A_1), \tag{14}$$

respectively.

Example 4.1. The interpolation formula (14), based on the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$$

for the function $F(A) = A^2$, has the form

$$L_1(F;A) = \frac{1}{2}\mathrm{tr}\, A \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

Example 4.2. Let $A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ be a functional matrix and

$$A_0 = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 0 \end{bmatrix}$$

be the interpolation nodes. Then

$$(A_0-A_1)\diamondsuit(A_0-A_1) = (A_1-A_0)\diamondsuit(A_1-A_0) = 2,$$

and the interpolation formula (14) takes the form

$$L_1(F;A) = \frac{1}{2}(x_{13}+x_{21}-3)F(A_0) - \frac{1}{2}(x_{13}+x_{21}-5)F(A_1),$$

and, therefore, we get that $L_1(F;A_0) = F(A_0)$, $L_1(F;A_1) = F(A_1)$.

Next, we consider a formula of the linear interpolation, similar to (7) and (11), with the multiplication in the case of Frobenius. We introduce the following notation:

$$\tilde{l}_{00}(A,B) = \frac{1}{D}\left[(A-A_1)\diamondsuit(B_1-B_2) - (A_1-A_2)\diamondsuit(B-B_1)\right],$$

$$\tilde{l}_{11}(A,B) = \frac{1}{D}\left[(A-A_0)\diamondsuit(B_2-B_0) - (A_2-A_0)\diamondsuit(B-B_0)\right],$$

$$\tilde{l}_{22}(A,B) = \frac{1}{D}\left[(A-A_0)\diamondsuit(B_0-B_1) - (A_0-A_1)\diamondsuit(B-B_0)\right],$$

where D is the numeric value, which is calculated by the formula

$$D = (A_0-A_1)\diamondsuit(B_1-B_2) - (A_1-A_2)\diamondsuit\diamondsuit(B_0-B_1).$$

The interpolation formula

$$\tilde{L}_{11}(A,B) = \tilde{l}_{00}(A,B)F(A_0,B_0) + \\ + \tilde{l}_{11}(A,B)F(A_1,B_1) + \tilde{l}_{22}(A,B)F(A_2,B_2) \tag{15}$$

satisfies the interpolation conditions $\tilde{L}_{11}(A_i,B_i) = F(A_i,B_i)$ $(i=0,1,2)$.

Example 4.3. Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be square matrices of the second order. We construct interpolation formulas of the form (7), (11) and (15) for the function $F(A,B)=(AB)^2$ on the nodes

$$A_0=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad B_0=\begin{bmatrix}0&2\\0&1\end{bmatrix};$$

$$A_1=\begin{bmatrix}1&2\\0&-1\end{bmatrix},\quad B_1=\begin{bmatrix}1&1\\1&2\end{bmatrix};\quad A_2=\begin{bmatrix}0&1\\1&2\end{bmatrix},\quad B_2=\begin{bmatrix}0&1\\2&1\end{bmatrix}.$$

In the case of the formula (7) we have

$$L_{11}[A,B]=\frac{1}{26}\times$$
$$\times\begin{bmatrix}-38+91a_{11}-156a_{12}+22a_{21}+123a_{22}+122b_{11}-144b_{12}+16b_{21}+246b_{22}\\1440+32a_{11}-388a_{12}+206a_{21}-192a_{22}+58b_{11}-328b_{12}+176b_{21}-378b_{22}\end{bmatrix}$$
$$\begin{bmatrix}-102+94a_{11}-16a_{12}-52a_{21}+126a_{22}+16b_{11}+8b_{12}-64b_{21}+168b_{22}\\128+211a_{11}-364a_{12}+138a_{21}+131a_{22}+82b_{11}-136b_{12}+24b_{21}+262b_{22}\end{bmatrix}.$$

Using the rule (11), we get that

$$L_{11}[A,B]=\begin{bmatrix}8-8a_{11}+4b_{11}&1+4b_{12}\\-11+11a_{21}+10b_{21}&7+4a_{22}-2b_{22}\end{bmatrix}.$$

Finally, for the formula (15) the value $D=-3$, and the required polynomial has the form

$$L_{11}[A,B]=$$
$$=\frac{1}{3}\begin{bmatrix}4(-4+2a_{11}-a_{12}+2a_{22}-b_{11}-2b_{12}+2b_{21}+5b_{22})\\-40+20a_{11}-21a_{12}+22a_{21}+20a_{22}+b_{11}-20b_{12}+20b_{21}+39b_{22}\end{bmatrix}$$
$$\begin{bmatrix}35-4a_{11}+4a_{21}-4a_{22}+4b_{11}+4b_{12}-4b_{21}-12b_{22}\\19+4a_{11}-14a_{12}+24a_{21}+4a_{22}+10b_{11}-4b_{12}+4b_{21}-2b_{22}\end{bmatrix}.$$

We note that all formulas, obtained in this example, have a different form, but for them the same interpolation conditions

$$L_{11}[A_0,B_0]=F(A_0,B_0)=\begin{bmatrix}0&9\\0&9\end{bmatrix},$$

$$L_{11}[A_1,B_1]=F(A_1,B_1)=\begin{bmatrix}4&5\\-1&-1\end{bmatrix},$$

$$L_{11}[A_2,B_2]=F(A_2,B_2)=\begin{bmatrix}8&5\\20&13\end{bmatrix}$$

are fulfilled.

5. Kronecker matrix multiplication and corresponding matrix polynomials

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are some matrices of the dimensions $m\times n$ and $p\times q$, respectively, then the Kronecker product of these matrices $A\otimes B$ is a matrix of dimension $mp\times nq$, which is defined by the formula

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{bmatrix}.$$

In general, the Kronecker product of matrices, in contrast to the Jordan multiplication, non-commutative, but has the property of associativity. The Kronecker multiplication is distributive with respect to the addition of matrices.

Let X be a set of square matrices, an operator $F:X\to Y$, where Y is also a set of square matrices of the fixed dimension, interpolation nodes $A_k\in X$ $(k=0,1,...,n)$ and there are inverse matrices $(A_i-A_j)^{-1}$ $(i\neq j)$. In addition, the dimension of matrices of the set Y coincides with the dimension of square matrices of the form $(A-A_\nu)\otimes I$.

We introduce the notation

$$l_k(A)=[(A-A_0)\otimes I]...[(A-A_{k-1})\otimes I]\,[(A-A_{k+1})\otimes I]...[(A-A_n)\otimes I].$$

Then for the polynomials

$$L_{0n}(A)=\sum_{k=0}^{n}F(A_k)\,l_k^{-1}(A_k)\,l_k(A), \tag{16}$$

$$L_{n0}(A)=\sum_{k=0}^{n}l_k(A)\,l_k^{-1}(A_k)\,F(A_k) \tag{17}$$

the equalities $L_{0n}(A_k)=L_{n0}(A_k)=F(A_k)$ are true, because

$$l_k^{-1}(A_k)\,l_k(A_\nu)=l_k(A_\nu)\,l_k^{-1}(A_k)=\delta_{k\nu}I.$$

Here and further the orders of matrices $F(A_k)$ are consistent with the order of the interpolation fundamental square matrices $l_k(A)$. If we select the expression

$$l_k(A)=[I\otimes(A-A_0)]...[I\otimes(A-A_{k-1})]\,[I\otimes(A-A_{k+1})]...[I\otimes(A-A_n)]$$

for the function $l_k(A)$ in (16) and (17), we come to some other kind of these formulas.

The formulas $L_{0n}(A)$ and $L_{n0}(A)$ are exact for the matrix polynomials

$$P_{0n}(A)=\sum_{k=0}^{n}B_k l_k^{-1}(A_k)\,l_k(A)\,,P_{n0}(A)=\sum_{k=0}^{n}l_k(A)\,l_k^{-1}(A_k)\,B_k,$$

where B_ν $(\nu=0,1,...,n)$ are arbitrary matrices from the set Y, respectively.

We consider formulas of the linear interpolation

$$L_{01}(A)=F(A_0)+[F(A_1)-F(A_0)]\left[I\otimes(A_1-A_0)^{-1}\right][I\otimes(A-A_0)],$$

$$L_{10}(A)=F(A_0)+[(A-A_0)\otimes I]\left[(A_1-A_0)^{-1}\otimes I\right][F(A_1)-F(A_0)].$$

The formula $L_{10}(A)$ is exact for matrix polynomials of the form $P_{10}(A) = A \otimes B + D$. Really,

$$L_{10}[P_{10}; A] = A_0 \otimes B + D + [(A - A_0) \otimes I] \left[(A_1 - A_0)^{-1} \otimes I\right] [(A_1 - A_0) \otimes B] =$$

$$= A_0 \otimes B + D + (A - A_0)(A_1 - A_0)^{-1}(A_1 - A_0) \otimes B =$$
$$= A_0 \otimes B + D + (A - A_0) \otimes B = P_{10}(A).$$

Similarly, the formula $L_{01}(A)$ is exact for matrix polynomials of the form $P_{01}(A) = B \otimes A + D$.

We consider the application of the Lagrange–Sylvester formula to construct the corresponding interpolation formulas, using several properties of the Kronecker multiplication for this. One of the important properties of this multiplication for the given problem is that the spectrum of the Cartesian product of matrices is clearly expressed through the spectrum of its factors.

Suppose that the matrix C has the form $C = A \otimes B$, and square matrices A and B of the orders p and q have the eigenvalues λ_i $(i = 1, 2, ..., p)$ and μ_j $(j = 1, 2, ..., q)$, respectively. Then [9] the matrix C has pq eigenvalues $\lambda_i \mu_j$ $(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$.

If the eigenvalues $\lambda_i \mu_j$ are different, then for the matrix C the Lagrange–Sylvester formula takes the form

$$F(C) = \sum_{k=1}^{p} \sum_{\nu=1}^{q} \frac{l_{k\nu}(C)}{l_{k\nu}(\lambda_k \mu_\nu)} F(\lambda_k \mu_\nu),$$

where

$$l_{k\nu}(C) = \prod_{i=1, i \neq k}^{p} \prod_{j=1, j \neq \nu}^{q} (C - \lambda_i \mu_j I_{pq}),$$

$$l_{k\nu}(\lambda_k \mu_\nu) = \prod_{i=1, i \neq k}^{p} \prod_{j=1, j \neq \nu}^{q} (\lambda_k \mu_\nu - \lambda_i \mu_j),$$

I_{pq} is the identity matrix of the pq-dimension.

We give the trigonometric variant of the Lagrange-Sylvester formula for the Kronecker product of matrices $C = A \otimes B$:

$$F(C) = \sum_{k=1}^{p} \sum_{\nu=1}^{q} \frac{\tilde{l}_{k\nu}(C)}{\tilde{l}_{k\nu}(\lambda_k \mu_\nu)} \times$$
$$\times \left(\frac{F(\lambda_k \mu_\nu) + F(-\lambda_k \mu_\nu)}{2} I_{pq} + \frac{F(\lambda_k \mu_\nu) - F(-\lambda_k \mu_\nu)}{2 \sin(\lambda_k \mu_\nu)} \sin C \right),$$

where

$$\tilde{l}_{k\nu}(C) = \prod_{i=1, i \neq k}^{p} \prod_{j=1, j \neq \nu}^{q} (\cos C - \cos(\lambda_i \mu_j) I_{pq}),$$

$$\tilde{l}_{k\nu}(\lambda_k \mu_\nu) = \prod_{i=1, i \neq k}^{p} \prod_{j=1, j \neq \nu}^{q} (\cos(\lambda_k \mu_\nu) - \cos(\lambda_i \mu_j)),$$

and I_{pq} is the identity matrix of the pq-dimension as before.

6. Infinite matrix and some interpolation formulas

Operators of the discrete convolution, as well as continuous, are widely used in the solution of many mathematical and applied problems [10–12]. Discrete convolutions can be applied to the interpolation problem of functions with many variables and infinite matrix variables.

Matrix $A = [a_{ij}]$ with real or complex elements a_{ij} is called infinite, if $i, j = 1, 2, ...$ or at least one of the indices i or j has infinite number of the values. Addition and multiplication of the infinite matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined the same way as in the finite-dimensional case. In contrast to the finite matrices, the product $AB = [c_{ij}]$ may not exist, since the series $c_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj}$ $(i, j = 1, 2, ...)$ may be divergent or nonsummable for all or only for the several i and j values. Moreover, if there is the existing product BA, the product AB may not exist. In general, the multiplication of infinite matrices is not associative: $(AB)\, C \neq A\, (BC)$.

On the set of infinite matrix A, on condition that the matrices A^k $(k \geq 2)$ exist, for entire functions $f(z)$ $(z \in \mathbb{C})$ the matrices $f(A)$ may be determined by the usual rules.

The theory of infinite matrices, as one of the sections of mathematical analysis, and its applications are interconnected with the theory of separable Hilbert spaces, including the coordinate Hilbert space l_2.

We consider some formulas for the interpolation of functions, given on the set of infinite sequences, which we denote by l. Each element x (infinite-dimensional vector) from l is defined by its coordinates: $x = \{x_k\}_{k=0}^{\infty} = \{x_0, x_1, x_2, ...\}$, where x_k $(k = 0, 1, ...)$ are complex numbers or complex random values with given distribution laws. Here the addition of elements of the set and its multiplication by a number are determined by the usual rules, and the product $x * y$ is given by the discrete convolution of the Laplace according to the rule

$$x * y = \left\{ \sum_{\nu=0}^{k} x_{k-\nu} y_{\nu} \right\}_{k=0}^{\infty};$$

the product $x * y$ also belongs to l. For this multiplication rule the sequence $I = \{1, 0, 0, ...\}$ is the unit, and in this case the set l is a commutative algebra.

Let F be operator, mapping the set l into l, and the elements $x_0 = \alpha_0 I$, $x_1 = \alpha_1 I$ and $x_2 = \alpha_2 I$, where I is the unit element in l, $\alpha_i \in \mathbb{C}$, $\alpha_j \neq \alpha_i$ for $j \neq i$ $(i, j = 0, 1, 2)$, are taken as the interpolation nodes. Then simplest on l formulas are formulas of the linear and quadratic interpolation

$$L_1(F; x) = F(x_0) + \frac{1}{\alpha_1 - \alpha_0} [F(x_1) - F(x_0)] * (x - x_0),$$

$$\begin{aligned} L_2(F; x) = {} & \frac{1}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)} F(x_0) * (x - x_1) * (x - x_2) + \\ & + \frac{1}{(\alpha_1 - \alpha_0)(\alpha_1 - \alpha_2)} F(x_1)(x - x_0) * (x - x_2) + \\ & + \frac{1}{(\alpha_2 - \alpha_0)(\alpha_2 - \alpha_1)} F(x_2) * (x - x_0) * (x - x_1), \end{aligned}$$

respectively, for which $L_1(F;x_0) = F(x_0)$, $L_1(F;x_1) = F(x_1)$ and $L_2(F;x_i) = F(x_i)$ $(i = 0,1,2)$.

For the same system of interpolation nodes $x_i = \alpha_i I$ on condition that $\alpha_j \neq \alpha_i$, $j \neq i$ $(i,j = 0,1,2,...,n)$, the Lagrange formula of the n-th order is written in the analogous form

$$L_n(F;x) = \sum_{k=0}^{n} \omega_{nk}(x) * F(\alpha_k I), \tag{18}$$

where

$$\omega_{nk}(x) = \frac{(x-\alpha_0 I)\,(x-\alpha_1 I)\cdots(x-\alpha_{k-1}I)\,(x-\alpha_{k+1}I)\cdots(x-\alpha_n I)}{(\alpha_k-\alpha_0)(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)},$$

I is the unit element of the algebra l. It's obvious that $L_n(F;x_k) = F(x_k)$ $(k = 0,1,...,n)$.

Let us consider a slightly different variant of (18). By $l^{m\times m}$ we denote the set of $m\times m$-matrices of the form $X = [x^{ij}]$, where x^{ij} are elements from l, i.e. $x^{ij} = \left\{x_k^{ij}\right\}_{k=0}^{\infty}$ $(i,j = 1,2,...,m)$. Here the operations of addition and multiplication of matrices by a number are ordinary, and the multiplication of matrices $X = [x^{ij}]$ and $Y = [y^{ij}]$ from $l^{m\times m}$ is carried out according to the rule:

$$C = X * Y = \left[c^{ij}\right],$$

where $c^{ij} = \sum_{\nu=1}^{m} x^{i\nu} * y^{\nu j}$, i.e. $x^{i\nu} * y^{\nu j}$ means the product of sequences $x^{i\nu}$ and $y^{\nu j}$ also in the sense of the Laplace convolution given above. This set of matrices with indicated rules of multiplication also form an algebra.

We consider the formula of the form (18), in which the interpolation nodes x_ν are $m\times m$-matrices

$$x_\nu = \begin{bmatrix} x_\nu^{11} & x_\nu^{12} & \cdots & x_\nu^{1m} \\ x_\nu^{21} & x_\nu^{22} & \cdots & x_\nu^{2m} \\ \cdots & \cdots & \cdots & \cdots \\ x_\nu^{m1} & x_\nu^{m2} & \cdots & x_\nu^{mm} \end{bmatrix} \quad (\nu = 0,1,...,n)$$

with the elements x_ν^{ij} from l. It is required of nodes x_ν that the matrices $x_\nu - x_k$ are reversible in the ordinary sense.

Let the interpolation nodes be matrices of the form

$$\tilde{x}_\nu = x_\nu I = \left[x_\nu^{ij}, 0, 0, ...\right] \quad (i,j = 1,2,...,m;\ \nu = 0,1,...,n).$$

Then for an operator $F : l^{m\times m} \to l^{m\times m}$ and the formula

$$\tilde{L}_n(F;x) = \sum_{k=0}^{n} \tilde{\omega}_{nk}(x) * F(\tilde{x}_k),$$

where

$$\tilde{\omega}_{nk}(x) = l_{k,0}(x) * l_{k,1}(x) * \cdots * l_{k,k-1}(x) * l_{k,k+1}(x) * \cdots * l_{k,n}(x),$$

$$l_{k,\nu}(x) = (x-\tilde{x}_\nu) * (\tilde{x}_k - \tilde{x}_\nu)^{-1} \equiv (x-\tilde{x}_\nu) * (x_k - x_\nu)^{-1} I \quad (k,\nu = 0,1,...,n)$$

the interpolation conditions $\tilde{L}_n(F;\tilde{x}_k) = F(\tilde{x}_k)$ $(k = 0, 1, ..., n)$ are fulfilled. These conditions take place by virtue of the equalities $\tilde{\omega}_{nk}(\tilde{x}_\nu) = \delta_{k\nu} I$, where, as before, $\delta_{k\nu}$ is the Kronecker symbol.

Example 6.1. Let A and B be infinite rectangular matrices of the dimensions $2 \times \infty$ and $\infty \times 2$, respectively:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \\ \vdots & \vdots \end{bmatrix}.$$

Their product is a (2×2)-matrix $AB = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where the elements S_{ij} $(1 \le i, j \le 2)$ are given by series

$$S_{11} = \sum_{i=1}^{\infty} a_{1i} b_{i1},\ S_{12} = \sum_{i=1}^{\infty} a_{1i} b_{i2},\ S_{21} = \sum_{i=1}^{\infty} a_{2i} b_{i1},\ S_{22} = \sum_{i=1}^{\infty} a_{2i} b_{i2}.$$

For the existence of the product AB it is required that these series are converging in some sense. For example, if the elements of matrix A and B are random values or processes, then one of the variants of the convergence may be the convergence of mathematical expectations of the summands of these series. We consider an example with this type of convergence.

Suppose that

$$a_{1i} = \frac{1}{(2i-1)!} W^{4i-2}(t),\ a_{2i} = \frac{1}{(2i-2)!} W^{4i+2}(t);$$

$$b_{i1} = \frac{(-1)^{1+i}}{[(4i-3)!!]^2} \xi^{4i-2}(t),\ b_{i2} = \frac{(-1)^{1-i}}{[(4i+1)!!]^2} \xi^{4i+2}(t),$$

where $W(t)$ is standard Wiener process, $\xi(t)$ is a random Gaussian process with zero mean value and variance $\sigma = \sigma(t)$. We assume that these processes are stochastically independent. We remind that the k-th moments of the processes $W(t)$ and $\xi(t)$ are given [13] by the equalities

$$E\left\{W^k(t)\right\} = \begin{cases} (2\nu-1)!!t^\nu, & k = 2\nu; \\ 0, & k = 2\nu+1, \end{cases}$$

$$E\left\{\xi^k(t)\right\} = \begin{cases} (2\nu-1)!!\sigma^\nu, & k = 2\nu; \\ 0, & k = 2\nu+1 \end{cases}$$

$(\nu = 0, 1, ...)$. In this case, the series $E\{S_{j\nu}\}$ $(j = 1, 2;\ \nu = 1, 2)$ converge. Since

$$E\{S_{11}\} = \sum_{i=1}^{\infty} E\{a_{1i} b_{i1}\} = \sin(t\sigma(t)),\ E\{S_{22}\} = \sum_{i=1}^{\infty} E\{a_{2i} b_{i2}\} = t\sigma(t)\cos(t\sigma(t)),$$

then the mathematical expectation of the trace of matrix AB has the simple form

$$E\{\mathrm{tr}(AB)\} = \sin(t\sigma(t)) + t^3\sigma^3(t)\cos(t\sigma(t)).$$

Construction and research of interpolation operator polynomials in the Hilbert spaces, which theory in some cases is interconnected with the infinite matrix theory, are considered in the articles [14–15].

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ON THE APPLICATION OF MULTIPARAMETER INVERSE EIGENVALUE PROBLEM AND NUMERICAL METHODS FOR FINDING ITS SOLUTION

O. S. Yaroshko

Резюме. У роботі здійснено огляд відомих прикладів практичних застосувань оберненої задачі на власні значення у різних наукових та інженерних сферах досліджень. Крім того, представлено існуючі чисельні методи та різноманітні техніки відшукання розв'язку оберненої спектральної задачі.

Abstract. This survey collects the known examples of practical application of inverse eigenvalue problems in different scientific and engineering areas. It also provides an overview of the existing numerical methods and different techniques for finding the solution of the inverse eigenvalue problem.

1. Introduction

An inverse eigenvalue problem is a subject of interest of different authors. There are numerous examples of practical application of this problem and of the analysis of its partial cases. In this article we try to make an overview of the most known and interesting examples of practical application of this type of problems.

Let $A(c)$ be an afinne family

$$A(c) = A_0 + \sum_{k=1}^{n} c_k A_k, \tag{1}$$

where $c \in R^n$, and $\{A_k\}$ are real symmetric matrices of dimension $n \times n$.

Let's also denote the eigenvalues of the matrix $A(c)$ as $\{\lambda_i(c)\}_1^n$, where $\lambda_1(c) \leq ... \leq \lambda_n(c)$.

The following problem is known as *the general inverse eigenvalue problem*:

Problem 1. *Provided real numbers $\lambda_1^* \leq ... \leq \lambda_n^*$ find $c \in R^n$ such that the eigenvalues of* (1) *satisfy the condition $\lambda_i(c) = \lambda_i^*$, $i = 1, ..., n$.*

One of the partial cases of the Problem 1 is *the additive inverse eigenvalue problem*:

Problem 2. *Let the linear family* (1) *be defined as $A_k = e_k e_k^T$, $k = 1, ..., n$ where e_k is a k-th unit vector such, that*

$$A(c) = A_0 + D, where D = diag(c_k) \tag{2}$$

Key words. Inverse eigenvalue problem; inverse spectral problem; Sturm-Liouville problem; eigenvalue; eigenvector; numerical method; iteration procedure; Newton-like methods.

Provided the real values $\lambda_1^* \leq ... \leq \lambda_n^*$ *find* $c \in R^n$ *such, that the eigenvalues of the matrix* (2) *satisfy the condition* $\lambda_i(c) = \lambda_i^*, i = 1, ..., n$.

Another partial case of general problem, that is considered in this survey, is *the multiplicative inverse eigenvalue problem*:

Problem 3. *Given a real symmetric matrix* A *and its eigenvalues* $\lambda_1^* \leq ... \leq \lambda_n^*$, *find an additive diagonal matrix* $D = diag(c_k)$, $c \in R^n$, *such that the result matrix* AD *has the given eigenvalues.*

Both additive and multiplicative inverse eigenvalue problems have been formulated by Downing and Householder (1956).

It is known that the inverse eigenvalue problems arise in different scientific areas, including systems of identification, seismic topography, geophysics, molecular spectroscopy, structural analysis, mechanic systems simulation and so on. Some of the partial cases of inverse eigenvalue problem appear in factor analysis, educational testing problem, etc (see [1] and the cited references).

2. Examples and practical application of the inverse eigenvalue problems

The classical example of inverse eigenvalue problem is the problem of finding a solution of inverse Sturm-Liouville problem. The continuous problem has been investigated by, for example, Borh, Gelfand, Levitan and Hald. The discrete analog can be found in the survey [3], a more detailed overview is presented below.

Let's consider a boundary problem [3]:

$$-u''(x) + p(x)\,u(x) = \lambda u(x),$$

$$u(0) = u(\pi) = 0.$$

The task is to find the potential $p(x)$ by using the given spectrum $\{\lambda_i^*\}_1^\infty$. In order to build the discrete analog, the authors [3] use a uniform mesh, defining $h = \frac{\pi}{n+1}$, $u_k = u(kh)$, $p_k = p(kh)$, $k = 1, ..., n$, and make a suggestion that the values $\{\lambda_i^*\}_1^\infty$ are known. By using the finite differences for the approximation u'', the following equation is received:

$$\frac{-u_{k+1} + 2u_k - u_{k-1}}{h^2} + p_k u_k = \lambda_j^* u_k, k = 1, ..., n, u_0 = u_{n+1} = 0,$$

where λ_j^* is an eigenvalue from the set $\{\lambda_i^*\}_1^n$.

Thus,it is obtained the additive inverse eigenvalue problem (2) with the matrix

$$A_0 = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & & 2 \end{pmatrix} \tag{3}$$

and $D = diag(p_k)$.

Another well known example is the inverse spectral problem which arises in the analysis of string vibrations. A reference to this example can be found, for example, in [1], [3]. Let's briefly explain the content of this problem.

Consider the corresponding boundary problem [3]:

$$-u'' (x) = \lambda p (x) u (x) ,$$
$$u (0) = u (\pi) = 0.$$

It is needed to find the density function $p (x) > 0$, under the condition that the fixed eigenvalues $\{\lambda_i^*\}_1^\infty$ are known. In order to proceed to the discrete analog of this problem, the transformations, similar to the case of Sturm-Liouville problem, are performed. As a result, the following equation is obtained:

$$Au = \lambda_i^* Du, i = 1, ..., n,$$

or, if reformulating a bit:

$$D^{-1}Au = \lambda_i^* u, i = 1, ..., n,$$

where $D = diag (p (kh)) > 0$, and the matrix A is defined by the correlation (3).

It can be easily seen that the obtained problem is the multiplicative inverse eigenvalue problem.

It is also possible to rewrite this problem in the form (1), where $A_0 = 0$, $A_k = e_k a_k^T$, $k = 1, ..., n$, and the a_k^T is a k-th row if the matrix A.

There are several inverse spectral problems with a matrix of a specific structure. For example, the problem of reconstructing the Jacobi matrix from the given spectral data. Briefly speaking, the inverse eigenvalue problem with the Jacobi matrix consists in defining the elements of the matrix from the given spectral data. This problem plays an important role in different applications, including vibration theory and structural design [10]. In some cases only a limited number of eigenvalues of the Jacobi matrix is provided. For example, four or five, as in the problem, presented in [10].

An interesting partial case of the general inverse spectral problem is the inverse Toeplitz problem (see [6]). According to the author, it is important, that although the Toeplitz matrices have such special structure, the question of solvability is opened for the case $n \geq 5$.

An inverse eigenvalue problem with a symmetrix matrix arises, for example, in the applied physics and the theory of control. This problem is investigated in the survey [9] and the cited references.

The other areas where the Problem 1 arises are nuclear spectroscopy and molecular spectroscopy. In practice the formulation of such problem often includes less parameters that there are eigenvalues. In such cases it makes sense to consider the problem formulation in least squares:

$$\min_{c \in R^n} \sum_{i=1}^{m} (\lambda_i (c) - \lambda_i^*)^2 .$$

An important type of problems arising in the engineer researches can be described with the following formula

$$\min\nolimits_{c \in R^m} f (c) \text{ by } l \leq \lambda_i (c) \leq u, i = 1, ..., n,$$

where $f(c)$ is a real-valued function of purpose, l and u are fixed lower and upper boundaries of eigenvalues of matrix $A(c)$, which is defined by the correlation (1). It's interesting to mention that the solution of the given problem often includes multiple eigenvalues, because the minimization of the function of purpose can simultaneously conduct several eigenvalues to the same boundary. This is why it's very important to choose the numerical method of solving the inverse spectral problem so that it correctly handles the case of multiple eigenvalues.

3. Numerical methods for solving the inverse eigenvalue problems

There is the rich literature dedicated to the question of numerical methods for finding an approximate solution of the inverse spectral problem. One of the creators of this theory is Friendland, who developed four quadratically convergent numerical methods together with his colleagues [3]. One of the methods, presented in [3], is, basically, the Newton method for solving the following system of nonlinear equations:

$$f(c) = \begin{bmatrix} \lambda_1(c) - \lambda_1^* \\ \ldots \\ \lambda_n(c) - \lambda_n^* \end{bmatrix} = 0,$$

where $\lambda^* = [\lambda_1^*, ..., \lambda_n^*]^T \in R$, and $\lambda(c) = [\lambda_1(c), ..., \lambda_n(c)]^T$ is the vector of unique eigenvalues of the matrix $A(c)$. Each $\lambda_i(c)$ is a real-valued function, differentiable in some neighborhood of the point c^*, if c^* is the solution of Problem 1.

Note, that each iteration of this method involves solving a full spectral problem for the matrix $A(c)$.

Two other methods from [3] are considered to be the modifications of the Newton method, where the calculation of eigenvectors is simplified. This means that instead of calculating the exact eigenvectors, or in other words, solving the corresponding spectral problem, the approximation of these eigenvectors is calculated. The fourth method from [3] originally is based on the work of Biegler-Konig, (see [4] and the cited references), and uses the idea of calculating the determinant.

Based on the methods developed by Friedland and others [3], there have been constructed new methods for solving some inverse eigenvalue problems by other scientists. For example, in the paper [6] there are presented two methods for finding the solution of an inverse singular problem: one of the methods is continuous, the other – discrete. The discrete method generalizes the iteration process, originally proposed by Friedland for solving an inverse spectral problem. The new method converges locally under the condition of existence of the problem's solution.

Different authors have investigated this methods. Ones of the firsts who used it, where Downing and Householder – for solving the additive and the multiplicative inverse spectral problems. For a long time this method was also used by the physics in the nuclear spectroscopy calculations.

Instead of calculating the exact eigenvectors of the matrix $A(c)$ on each iteration of the method, it is possible to approximate them by using, for example, the inverse iteration. On this idea the Method II [3] is based.

The Method III is built on the idea of using a matrix of exponentials and the Cayley transform.

As explained by the authors in the survey [6], from the geometric point of view, the Method III [3] can be interpreted as the classical Newton method. This means that the geometry which is involved in the Method III, is closely bound to the geometry of the Newton method for the nonlinear equations with one variable. Consequently, the Method III can be generalized to the iteration process for calculation the approximate solution of the inverse singular problem.

Investigation of the methods, described in [3], can be found in other various articles, for example – in [1]. As it is stated by the author [1], in case of a matrix of big dimensions, the Method III has an obvious disadvantage: constructing an inverse matrix on each step is an expensive operation. These expenses can be decreased by using the iteration procedures (inner iterations). Because of it, usually the Method III, as the other methods of this type, is too expensive in such sense that the number of performed iterations (inner iterations) is much bigger then the number of iterations needed for convergence of the Newton method (outer iterations).

In order to calculate the solution of the classic additive and multiplicative inverse eigenvalue problems the Newton-like methods are also fine to use.

Among the known methods of this type it is worth mentioning the algorithm suggested by Kublanovskaya [2]. This algorithm calculates the solution as a zero of the function

$$H(c) = \begin{bmatrix} \lambda_1(c) - \lambda_1 \\ \vdots \\ \lambda_n(c) - \lambda_n \end{bmatrix},$$

where $\lambda_1(c) < ... < \lambda_n(c)$ are the eigenvalues of the matrix $A(c)$, and $\lambda_1 \leq ... \leq \lambda_n$ are the given eigenvalues.

As an alternative to the Kublanovskaya method, there is another algorithm presented in [2]. This one is also a Newton-like method and it calculates the solution of the initial problem, as the zeros of the function

$$F(c) = \begin{bmatrix} det(A(c) - \lambda_1 I) \\ \vdots \\ det(A(c) - \lambda_n I) \end{bmatrix}.$$

In order to reduce extra expenses of the exact iteration methods and to increase the effectiveness, the scientists Chan, Chung and Xu (see [1] and the cited references) suggested in inexact Newton-like method, which is used for the matrices of big dimensions. The inexact Newton method stops the inner iteration process before it converges. Thus, it is possible to decrease the total number of both, inner and outer, iterations, by choosing a proper stop condition.

In the paper [1] another approach is put forward – an inexact method of Cayley transform for the inverse eigenvalue probelm. This methods also minimizes the extra expenses and increases the productivity.

Based on the differentiation theory and on the QR-decomposition of a matrix, Li suggested a numerical method for solving the inverse spectral problems, which works for the case of unique eigenvalues (see [4] and the cited references).

In the same paper [4] there is examined the formulation and local convergence of a quadratically convergent method for solving the general inverse eigenvalue problem provided that its solution exists. The proposed method is based on the mentioned QR-decomposition of a matrix and the ideas of Li and Dai (see [4] and the cites references). As it is stated by the authors, this method is applicable for the case of unique eigenvalues as well as for multiple eigenvalues of the matrix.

One more approach to building a numerical method for solving an inverse spectral problem is suggested in the survey [9]. This approach is based on the analysis of analyticity of eigenvalues and eigenvectors of matrix of the problem. The examination of analyticity of spectral problems has a long history (see [9] and the cited references). However, according to the author, relatively small attention has been paid to the examination of analyticity of matrix spectra in the case when the matrix analytically depends on several parameters. Thereby, in [9] a new method is proposed. This is another modification of the known Newton method and allows to find the approximate solution of an inverse eigenvalue problem with a real symmetric matrix, which depends on several parameters.

Recently another approach type of methods – the gradient methods – gained the attention of scientists. For example, a variation-gradient method for solving multiparameter eigenvalue problems has been developed by Klobystov and Podlevkyi (see [5], [7]). The proposed method was later modified and extended to the inverse spectral problem by Podlevskyi and Yaroshko (see [8]). The idea of these methods, for both direct and inverse multiparameter eigenvalue problems, is to replace the spectral problem with an equivalent variation problem and applying the iterative method to find the solution of this variation problem. The mentioned method is based on the gradient procedure and the Newton method.

Let's consider the following multiparameter spectral problem in the Euclidian space E^n:

$$T(\lambda)\,x \equiv Ax - \lambda_1 B_1 x - \cdots - \lambda_m B_m x = 0, \tag{4}$$

where $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ – are spectral parameters, $x = (x_1, \ldots, x_n) \in E^n$, and $A, B_1, ..., B_m$ – are some linear operators that act in the Euclidian space E^n.

Let's place in correspondence to the spectral problem (4) the variation problem of minimization of a functional:

$$F(u) = \frac{1}{2}\left\|T(\lambda)\,x\right\|_H^2, \forall u = \{x, \lambda\} \in H. \tag{5}$$

The problem (5) consists in finding such set of parameters $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ and the corresponding vector $x \in E^n \setminus \{0\}$ on which the functional $F(u)$ reaches its minimal value:

$$F(u) \to \min_u, u \in U \subset H, \tag{6}$$

where U is a set of points $u = \{x, \lambda\}$, that satisfies the equation (4), H is an Euclidian space.

It can be shown that the spectral problem (4) and the variation problem (6) are equivalent. This means that each eigenpair $\{x, \lambda\}$ of the problem (4) is a point of minimum $u = \{x, \lambda\}$ of the functional (5), and vice versa.

This result allows us to build the gradient procedure for the numerical solving of the problem (6) and, therefore, the problem (4):

$$u_{k+1} = u_k - \gamma(u_k) \nabla F(u_k), \quad k = 0, 1, 2, ... \tag{7}$$

The formula (7) describes the whole class of methods, which differ one from another only by the choice of the step $\gamma(u_k)$.

In our method we suggest calculating the value $\gamma_k = \gamma(u_k)$ on each step of the iteration process by the formula:

$$\gamma_k = \frac{F(u_k)}{\|\nabla F(u_k)\|_H^2}.$$

To conclude, the iteration process can be written in the form:

$$u_{k+1} = u_k - \frac{F(u_k)}{\|\nabla F(u_k)\|_H^2} \nabla F(u_k). \tag{8}$$

So far we have described the method for solving the direct eigenvalue problem. Let's explain the algorithm of solving the inverse spectral problem, which is based on the described gradient procedure.

Consider the inverse eigenvalue problem (1) with the real matrices A_0, $A_1, ..., A_m \in E^{n\times n}$, and where the pairs $\{\lambda_k, x^k\}_{k=1}^m$ are the eigenpairs of the matrix $A(p)$. Here $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$, $x^k \in H = E^n \setminus \{0\}$, $k = 1, ..., m$, and E is a real Euclidian space.

Using the definition of the eigenvalue and the corresponding eigenvector, we can build the system of m equations for finding the parameters $p_1, ..., p_m$:

$$\begin{cases} ((A_0 - \lambda_1 I) + p_1 A_1 + ... + p_m A_m) x^1 = 0, \\ \qquad \cdots \\ ((A_0 - \lambda_m I) + p_1 A_1 + ... + p_m A_m) x^m = 0. \end{cases} \tag{9}$$

Now lets transform this system by introducing the matrix operators $\mathbf{A}, \mathbf{B}_i :$ $\mathbf{H} \to \mathbf{H}$, $\mathbf{H} = \oplus_{k=1}^m E^{n\times n}$ $(i = 1, ..., m.)$,

$$A = \begin{pmatrix} (A_0 - \lambda_1 I) & & 0 \\ & \ddots & \\ 0 & & (A_0 - \lambda_m I) \end{pmatrix}, \quad B_i = \begin{pmatrix} -A_i & & 0 \\ & \ddots & \\ 0 & & -A_i \end{pmatrix},$$

In case $\mathbf{x} = \left(x^1, x^2, ..., x^m\right)^T \in \mathbf{H}$, we get

$$\mathbf{A}\mathbf{x} = \left((A_0 - \lambda_1 I) x^1, (A_0 - \lambda_2 I) x^2, ..., (A_0 - \lambda_m I) x^m\right),$$

$$\mathbf{B}_i\mathbf{x} = \left(-A_i x^1, -A_i x^2, ..., -A_i x^m\right).$$

Now it is possible to pass from the problem (9) to the problem in the form (4) in the space $\mathbf{H}$

$$T(p) \equiv \mathbf{A}\mathbf{x} - p_1\mathbf{B}_1\mathbf{x} - ... - p_m\mathbf{B}_m\mathbf{x} = 0. \tag{10}$$

Therefore, we retrieved the problem of finding the set of parameters $p_1, ..., p_m$, such that the equation (10) has a non-trivial solution $\mathbf{x} \in \mathbf{H} \setminus \{\mathbf{0}\}$.

In correspondence to the problem (10) we put the variation problem:

$$F(\mathbf{u}) \to \min_{\mathbf{u}}, \mathbf{u} \in \mathbf{U} \subset \mathbf{H},$$

where $F(\mathbf{u}) = \frac{1}{2}\|T(p)\mathbf{x}\|_H^2, \quad \forall \mathbf{u} = \{\mathbf{x}, p\} \in H = \mathbf{H} \oplus E^m$.

As expected, the task is to find the set of parameters $p = \{p_1, ..., p_m\} \in E^m$ and the corresponding vector $\mathbf{x} \in \mathbf{H}\setminus\{\mathbf{0}\}$, on which the functional $F(\mathbf{u})$ reaches its minimal value.

In order to solve the variation problem we use the iteration process (8). Consequently, we obtain the solution of the initial inverse eigenvalue problem.

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