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Журнал публікує оригінальні та обзорні статті, дає інформацію про наукові конференції та семінари з проблем прикладної математики, кібернетики та системного аналізу, дає інформацію про новітні досягнення вітчизняної та закордонної науки по проблемам теорії систем і інформатики. Містить роботи з теоретичних проблем кібернетики, проблем штучного інтелекту, теорії оптимального керування та прийняття рішень. Публікуються роботи, присвячені питанням теорії моделювання динамічних систем, зокрема, теорії нелінійних систем, теорії стійкості руху. Розглядаються питання розробки програмно-технічних комплексів, математичного та програмного забезпечення, нових інформаційних технологій.

Для студентів, аспірантів та науковців, які працюють у галузі прикладної математики, кібернетики, інформатики.

Журнал публикует оригинальные и обзорные статьи, предоставляет информацию о научных конференциях и семинарах по проблемам прикладной математики, кибернетики и системному анализу, дает информацию о новейших достижениях отечественной и зарубежной науки о проблемах теории систем и информатики. Содержит работы по теоретическим проблемам кибернетики, проблемам искусственного интеллекта, теории оптимального управления и принятия решений. Публикуются работы по вопросам теории моделирования динамических систем, в частности, теории нелинейных систем, теории устойчивости движения. Рассматриваются вопросы разработки программно-технических комплексов, математического и программного обеспечения, новых информационных технологий.

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The journal publishes original and overview articles, provides information on scientific conferences and seminars on the problems of applied mathematics, cybernetics and system analysis, provides information on the latest achievements of domestic and foreign science about problems in the theory of systems and informatics. Contains works on theoretical problems of cybernetics, problems of artificial intelligence, theory of optimal control and decision-making. Works on the theory of simulation of dynamic systems, in particular, the theory of nonlinear systems, the theory of the stability of motion, are published. Questions of development of software and hardware complexes, mathematical and software, new information technologies are considered. For students, graduate students and scientists who work in the field of applied mathematics, informatics and cybernetics.

For scientists, professors, graduate and postgraduate students.

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ЗМІСТ

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СОДЕРЖАНИЕ

UDC 658.5.011

N. V. Bojko, PhD student
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

TECHNOLOGICAL ASPECTS IN RESEARCH ON EMERGING AND DEVELOPING OF CONFLICTS IN IT-PROJECTS

This paper is dedicated to the research on emerging the conflict situation in IT project management, leading to negative consequences during its implementation or development. The types of conflicts have been studied as well as differentiation of conflicts, project team management and project conflict management.

Originality of the paper is due to new approach on the conflicts in IT projects, highlighting the process of conflict removal at the emergence stage as well as in their development.

Keywords: conflict, project, management, management technologies.

Introduction. Many times domestic and foreign scientists have considered the problem of solving the conflicts and management of conflict situations in the organizations and offered various ways to solve them.

However, these approaches to solving the conflicts do not fit to solving the conflicts and conflicting situations in the projects.

The relationship among the group of persons working on the project are usually characterized by the collaboration and healthy competition. Persons depend on one another, while at the same time having personal interests, goals and abilities within a project. While on the project, persons tend to use acquired experience, obtain new knowledge, making a reputation of themselves and get the ability for career advance via getting responsible job assignments. Of importance for the future life is also new useful personal connections.

Every person encounter various conflicts in everyday life, in the family, on the job, anywhere with human factor involved. Various conflicts happen in different situations and lead to different consequences. Some conflicts invoke light annoying or disappointing emotions, the others tend to result in more drastic consequences (stress, depression, frustration).

Actuality. Conflicts in the working teams are practically inevitable; they are creating the working atmosphere both in production and development environments. However, not all conflicts have positive influence on the persons, groups and collectives. More often, the problems and stressed relations emerge between team members and involved individuals as the project systems are more complex. Preservation of friendly relations and preventing the inevitable misunderstandings and misinterpretations are of utmost importance in the projects.

Working at the project differs from working at the facilities by its short duration and predetermined value. Projects are limited in time and resources. Because of this, the manager solving of the conflicts in the project had to get the things right and make correct choices to amend the conflict.

Purpose of the paper is to analyze the conflicts and formation of proposals concerning the application of methods to solve conflicts in a specific situation.

Primary result. Any project begins as friendly relations, common and mutual understanding of a goals, project aim, responsibility, kindness towards coworkers and project members and other. Speaking professionally, one can say these stages of project lifecycle are called initiation or planning.

However, at the stage of implementation, monitoring and evaluation a minor communication difficulties appear between the customer, manager, team and other interested parties.

All these misunderstandings are transformed into questions, which remain unsolved by the parties where they originate. The questions are accumulated, get deeper and are transformed into misunderstandings. Persons at the stage of misunderstanding could painlessly solve them, but usually they do not want to or do not have enough time (or so they think) to solve these minor issues. Time passes in misunderstanding transform into offences and problems.

This is the edge traversed by information from simply unnoticed into a real problem. After this, problem can only be solved with the help from project management or intermediator. It is also possible that the problem and conflict will not be solved during the project implementation up until its closure.

Let us consider some definitions of a conflict.

Cognitive definition: conflict is the collision of different thinking types, each one aspiring to be representative.

Reflective definition: a conflict is the situation, which has the possibility for a deeper research of object and its environment and then it is possible to investigate on own thinking and get understanding why the thoughts about facts and problems are different.

Interactive definition: Conflict is a process of development of subject interaction from confrontation to communication.

The cognitive definition describes intellectual part of a conflict. Reflective definition is based on analysis of every component of the conflicting situation. Interactive definition stresses upon the specifics of interaction during different stages of conflict [1].

There are options concerning the borders of conflict, which are considered in the spatial, timewise and intrasystem aspects.

Spatial aspect denotes the development of a conflict on a given territory. Strict definition of a spatial borders of conflict is primary important in the international relations, which are closely tied to the problem of conflicting participants.

Timewise aspect define the conflict duration. The beginning of the conflict is specific point in time when both sides of the conflict start to consciously counter each other. The opposing positions by themselves are not yet a conflict, but the conflicting questions are starting to appear.

The intrasystem aspect. Each conflict is happening within some system, in the family, between coworkers, between groups of coworkers, in the regions, countries, or in the international community. The conflict between parties that are part of the same system can be private, deep, habitual or limited, but it all depends on the conflicting parties [2].

The conflict is not always causing damage and not necessarily involves negative outcome. The conflict contains in itself potential positive possibilities, but how to use them is not a common knowledge for those participating in it. Conflict – change

– adaptation – survival – conclusions – experience [3]. At the same time, a conflict can be a tool for diagnostics of collective state, its development, demise or advance, and a way of proliferation [4].

The source of every conflict is a contradiction, and the contradiction appear every time there is an inconsistency: of goals, interest, positions, thoughts, views, beliefs, personal qualities, interpersonal relations, knowledge, skills, abilities, management functions, finances, methods of action, motivations, needs, values, understandings: interpretation of information, evaluations and self-evaluations [5].

Let us consider the systematcs of a conflict, which appear in psychology, and conflictology. This is important for working relations regardless of whether the ownership of enterprise is private or public, domestic or foreign; conflicts among persons or groups, between person and a group or even internal. Let us also define the causes and problems of conflicts, styles of behavior, stages of conflict development and then ways to their resolving.

Now we outline which conflicts can appear in the project. The most popular conflict are the following: interpersonal and between person and a group (this is especially true for long-term or large-scale projects). Internal conflict of a person and conflicts between groups are of lower incidence (fig. 1).

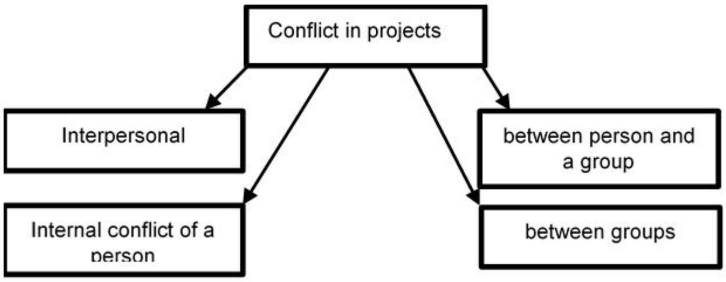

Fig.1. Conflicts in a project

Let us consider some types of conflicts.

Interpersonal conflict. This type of conflict can have different subtypes. However, with all collected and processed information we can make a conclusion that interpersonal conflicts can be divided into professional and social.

Professional conflicts are based on unsolved, not entirely understood questions of business character. Typically, persons holding the same position or being in the same team are less susceptible to conflicts. The possible reason for that is that persons who are working on the same administrative level and whose professional interests are not intersecting are less connected. However, there are exceptions to this rule. For example, when coworkers are working in same team and the same level, they can begin a business conflict among themselves if one person has an inherent conflicting attitude with or without reasons, or if their work is mutually dependent.

Interpersonal conflicts are more frequent between project management and its subordinate workers. The manager gives the task to do in the specific period for a worker. The subordinate completes the task, but the manager do not like the quality off the completed task. There are several approaches to a conflicting situation, one of which is when a worker did not pay attention to the pertinent information or did not complete the task to its final. Another approach is when the manager did not voice his thoughts correctly and the worker did not what was expected of him. In the first cause, the blame is on the subordinate. If this situation persists, the project manager should think if it is justified to keep the worker on the project. In the second cause, the project manager is to blame. The question is how the subordinate worker should behave to avoid unnecessary stress and conflict. A worker may ask additional questions to project manager in order to clear any inconsistencies. By getting answers from the project manager, a worker will be able to complete the task in the set timeframe and at the intended quality level.

The social conflicts. This type of conflict can arise at any area of activity and between different persons. The social conflict are the most misunderstood because they can appear out of nowhere and they do not have a constructive substantiation. Typically, the persons involved cannot explain why they started to conflict among themselves.

It is very hard or practically impossible to find a causing reason for social conflict. The following is a list of possible causes of the social conflicts:

1. The person started the conflict because he is in a bad mood;
2. The personality specifics, when a person believes that his solution is the only right one.
3. Loud tone of conversation such as offense, misunderstanding, insults;
4. Transmission of corrupted information;
5. Personal dislike of a coworker.

Typically, the persons involved did not want to solve the conflict by themselves. They may avoid each other try to adapt to the conflict or even find a ad-hoc settlement solution. But all of this are temporary means and the conflict do not disappear.

In this case, there are several possible options to solve the social conflict:

1. Finding a solution to the conflict on the will of one or both of the persons involved;
2. Involvement of the project manager to help solve the conflict;
3. Involvement of the intermediate or third party responsible for maintaining the positive climate in the collective.

The solution of a conflict by the persons involved is only possible if one or both persons are tired of spending energy to unwarranted efforts and the other side of a conflicts support this initiative; or when both conflicting persons assemble and sort out all problematic questions accumulated between them. This way of solving the conflict requires least efforts because there are no third parties involved.

Involvement of a project manager may solve the conflict, generate a new one or turn the existing conflict from private to open. It can only be solved if the project manager is limiting the communication between conflicting persons by transferring one of them to another job or position. If a project manager tries to solve the conflict forcefully, this may lead to conflict changing its form, to extend, to cause stress in the persons involved to provoke person to quit, or to degrade the quality of the work.

The best way would be an involvement of intermediator or third party who is responsible for the positive climate in the project if a project has such position. Any coworker or even manager of the project can be such an intermediator. The intermediator do not choose sides, and do not protect anyone's interests. He will necessarily ascertain the agreement from conflicting parties to provide help while keeping neutrality. This way is the most practical to solve the social conflict provided there are will to do it.

The conflict between a person and group of persons. This conflict arises when there are some staff changes during project development. This happen when a person quits, moves, takes long sick leave or has found a better job. Another worker is hired for his position. The existing team of the project do not want to accept a new coworker. Reasons for this are typically professional, or a new person may not be able to maintain the quality of his work, or show a certain annoying behaviors, or being just generally impolite towards his colleagues.

If a team do not accept a new coworker for a substantiated reason then, the project manager must solve the question how to continue the work by himself. Typically, the persons who have not gained an acceptance in the project will quit because of a pressure.

The less frequent but deserving consideration are intrapersonal conflicts and conflicts between groups.

Intrapersonal conflict. This is the most secretive conflict, almost impossible to detect. The project manager should keep a constant observation over a team and analyze the behavior of each team member. This is not usually possible because the project manager is the one tasked with the strategic matters.

Such a conflict may show itself in the form of a depression or stress and may lead to a lowering the quality of work or productivity of a worker. Person may solve such a conflict by himself if he gets a healp from the colleagues or the project manager.

The conflict between groups of persons. Such a conflict arise in the large-scale and long-time projects. The basis for such conflict can be: resources in their limited availability; different approaches to methods to complete the project; mutual dependence of the teams; administrative reasons.

In such projects, the project manager plays the role of an arbiter. He constantly solve the conflicting questions.

Let us consider the traditional methods of countering the conflicts. There are many classical ways to counter the conflicts. They are used in any area of human activity and in any types of enterprise. The most popular of them include: administratively enforcing; testing; intermediation; ignoring (fig. 2).

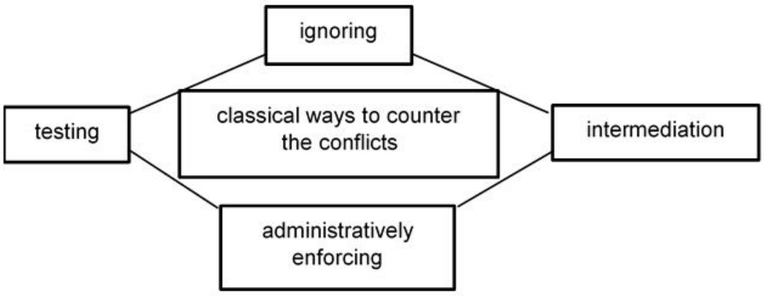

Fig. 2. Ways to counter the conflicts

The way of administrative enforcing. Often, the management uses this way, but it is not effective. In the conflicting situation, the manager acts as a force, which quickly stops the conflict. However, the conflict is not going away, it just changes shape and may reappear again later even stronger. In this case, the conflicting sides stop communicating. Nonetheless, the conflict itself as well as its reason, remain.

When the conflict undergo transformation and reappear stronger, the management may not be able to handle it again. As a result, such conflict may lead to negative consequences for an entire project. The project may be postponed for an indefinite period, or a persons involved will voluntarily quit, or the pressure upon the rest of the team will be too much and the implementation of the project will stop completely.

After the management uses this way, the persons or sides may remain offended by his behavior and start to complete their work less enthusiastic and with lower quality. As a result, this will degrade overall quality of the project. The worker may quit with from the project without explaining his decision. The project manager will then acquire the reputation of incompetent leader, and the worker enrolling on different projects will try to avoid working together with such a manager. The person will always remember such behavior of the manager and will remain offended.

This way is even more negative because the reason of the conflict between persons is not investigated. The project manager should remember that using this way, he would inevitably cause disapproval from the team and unwanted consequences for the project.

The testing way. Using this way as a counter to the conflicts in the project may lead to a partial solving of the conflict. It is only effective when it is necessary to establish a possible reasons or causes of the problem leading to the conflict.

If the test is done right, it is possible to get an information about possible reasons of the conflicting situation. Information is being collected, analyzed and shown to the project manager.

Testing information is only useful if the test subjects maintain honest answers. If the answers were not true and honest, the test will yield corrupted information.

The possible course of action for the project management are the following: he can ignore the information collected; he can assemble meetings or conduct interviews with the subordinate managers or persons who responded negatively on the test, talk to them privately, change the tactics of communication, conduct encouraging events (holidays, celebrations, games) etc.

The project manager decides upon using one of the aforementioned options. The practice shows that the statistical information is typically ignored, so the emergence of the conflict in the project is generally possible.

The intermediation. This way can solve the conflict and find out its causes, provided it is used skillfully and correctly. This is very hard to achieve because there are many influencing factors such as mentality education, personal preferences and experience.

The intermediator shall have good basics in psychology or be generally experienced. Better if he had experience of solving the conflicts this way. If the intermediation is enforced to solve the conflict, it may actually worsen it. It is very important that the conflicting sides should voluntarily agree to solve the conflict this way.

Typically, the person acting as an intermediator act incorrectly by supporting interests of one of the conflicting sides. This may be evident in his behavior or conversation.

However, if one is to keep an established behavioral norms and general rules, it is entirely possible to find out the causes of the conflict and problem leading to it.

Way of ignoring. This way is typical for short-term projects, such as IT projects, but is also found in the medium-term or long-term projects. In IT projects, the project manager may not pay attention to the conflict because the typical project lasts from 3 to 6 months. After the project is completed, the conflicting sides may not work on another projects or generally part their professional ways.

The way of ignoring maybe the part of life for a project manager, whose creed is “I don't see it therefore it doesn't exist”. If the project manager is always using this way then the conflicting sides themselves will solve the conflict or the conflicts will overcome the project and it will be cancelled or completed past deadline and quality requirements.

All aforementioned ways of dealing with the conflicts are found in solving the conflicts in projects; also some methods may be used together or change shape.

The professional manager will not give orders, ignore or be intermediator in the conflict. He will try to spot a problem as it arises, and begin to use methods of nonforceful nature to amend the person's attitude towards the problem.

Conclusions.

The analysis shown in this paper confirms that the problematics of solving the conflicts and managing conflicting situations in the projects is not highlighted nowadays. The author have analyzed the conflicts which appear in projects and ways to counter them. The originality of the work is warranted by the new approach on the conflict in the projects, isolating the processes of interactions as instrumental on the way to counter such conflicts. The reasons for the emergence of the conflicts, general classification of the conflict, stages of the conflict, the traditional ways of countering the conflicts, and approaches of the project management are listed in this article.

References:

1. Модели и методы управления персоналом: российско-британское учеб. пособ. для магистерских спец. Московской высш. шк. соц. и экон. наук. Серия "Библиотека журнала "Управление персоналом" / под ред. Е. Б. Моргунова. – М. : ЗАО "Бизнес-школа", "Интел-Синтез", 2001. – 464 с.
2. Кіжнер Л. О. Конфліктологія : навч. посіб. / Л. О. Кіжнер, М. В. Мартиненко, О. В. Мангольд. – Х. : ХНЕУ, 2007. – 144 с.
3. Гольфельд Я. А. Универсальность в разрешении конфликта : практико-ориентиров. методика по курсу "Конфликтология" / Я. А. Гольфельд // Междунар. науч. шк. универсологии. Серия "От конфликта к гармонии". – Луганск, 2003. – Кн. 1. – 164 с.
4. Владимиренко В. Є. Філософія попередження і розв'язання конфліктів : метод. посіб. для ділових людей та студ. спец. "Математика і економіка" / В. Є. Владимиренко. – К. : Аваль, 1998. – 41 с.
5. Курбатов В. И. Конфликтология : учеб. пособ. для студ. гуманит. вузов. Серия "Высшее образование" / В. И. Курбатов. – Ростов н/Д : Феникс, 2005. – 445 с.

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Н. В. Бойко, асп.
Київський національний університет імені Тараса Шевченка, Київ, Україна

ТЕХНОЛОГІЧНІ АСПЕКТИ ДОСЛІДЖЕННЯ ВИНИКНЕННЯ Й РОЗВИТКУ КОНФЛІКТІВ У ІТ-ПРОЕКТАХ

Досліджено причини виникнення конфліктної ситуації в управлінні ІТ-проектами, які призводять до негативних наслідків під час реалізації проекту. Розглянуто типи конфліктів, розмежування конфліктів, управління командами проектів, конфліктами проектів. Оригінальність статті полягає в новому погляді на конфлікти в проектах, виділенні процесів їхнього усунення як на стадії виникнення, так і в процесі розвитку.

Ключеві слова: конфлікт, проект, управління, технології управління.

Н. В. Бойко, асп.
Киевський национальний университет имени Тараса Шевченка, Киев, Украина

ТЕХНОЛОГИЧЕСКИЕ АСПЕКТЫ ИССЛЕДОВАНИЯ ВОЗНИКНОВЕНИЯ И РАЗВИТИЯ КОНФЛИКТОВ В ИТ-ПРОЕКТАХ

Исследованы причины возникновения конфликтной ситуации в управлении ИТ-проектами, которые приводят к негативным последствиям во время реализации проекта. Проведено исследование типов конфликтов, разграничение конфликтов, управления командами проектов, конфликтами проектов. Оригинальность статьи заключается в новом взгляде на конфликты в проектах, выделении процессов их устранения как на стадии возникновения, так и в процессе развития.

Ключевые слова: конфликт, проект, управление, технологии управления.

УДК 579.929.4

A. S. Bychkov, PhD in Math
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

MODELS AND METHODS FOR PULSED CONTINUOUS-DISCRETE INFORMATION SYSTEMS AND PROCESSES INVESTIGATION

The article deals with the stability of a new class of continuous-discrete information systems –fuzzy linear hybrid pulsed automata. Sufficient conditions of stability are obtained. Stability and stability on the part of variables are proved. The cyclic cases are described. Research is conducted with method of Lyapunov functions.

Keywords: information systems and processes, Lyapunov method, stability, hybrid automata, L-stability.

Introduction

Nowadays more and more problems appear that require sophisticated mathematical models for adequate modeling. Using these models leads to substantial problems with investigating of their behavior. Decomposing these models into simpler ones and moving to continuous-discrete systems, we get a new methodology of research.

This work applies Lyapunov functions method applies to fuzzy cyclic hybrid automata. Stability of hybrid automatawas studied by many authors [1-6]. In [7,8] the approach does not require finding solutions of hybrid automaton. This approach is developed to study the stability of fuzzy pulsed hybrid automata. Fuzzy is considered in terms of [9,10].

Key definitions and preliminary results

We introduce the following notation.

$\|\cdot\|$ is a Euclidean norm in R^d space(notation is the same for all d);

$$\nabla_f V(y_0) = V'(y_0) f(y_0) \text{, if } V:\mathrm{R}^d \to R \text{ i } f:\mathrm{R}^d \to \mathrm{R}^d \text{ or } f:\mathrm{R}^d \to \mathrm{R}^{d\times l}.$$

Definition 1. Fuzzy hybrid automaton with fuzzy switching (FHAFS) is a tuple $HA = (Q, Y, PS, g, w, h, Inv, Init, Jump)$ where

- Q is a finite set of discrete states;
- $Y = \mathrm{R}^d$ – a set of continuous states;
- $PS = (X, 2^X, P)$ – space of capabilities with the normalized degree of possibilities;
- $w:\mathrm{R}^+ \times X_+ \to \mathrm{R}^l$ – process of fuzzy walk in space PS;
- $g: Q\times \mathrm{R}^d \to \mathrm{R}^d$, $h: Q\times \mathrm{R}^d \to \mathrm{R}^{d\times l}$ – partly defined functions that define continuous behavior of automaton while staying in the discrete state; we'll call them $g_q = y \mapsto g(q,y)$ and $h_q = y \mapsto h(q,y)$;
- $Inv: Q \to Y \setminus \{\varnothing\}$ – function that defines a set where discrete state is invariant;
- $Init \subseteq Q\times Y$ - a set of initial conditions;
- $Jump: Q\times Y \times X \to 2^{Q\times Y}$ – map that specifies the transition between discrete states;

and the condition $Inv(q) \subseteq \operatorname{Dom} g_q \cap \operatorname{Dom} h_q$ holds for all $q \in Q$.

Definition 2. Phase orbit that is implemented by FHAFS HA is a map $\chi = (\tau, \overline{q}, \overline{y}, x)$ where $x \in X_+$, $\tau = (I_i)_{i=0}^N \in \mathrm{H}T$, $\overline{q}: \langle\tau\rangle \to Q$ is a map, and $\overline{y} = (y^i)_{i\in\langle\tau\rangle}$ is an indexed family of continuous maps $y^i: I_i \to Y$ such that:

1) $y^i(t) \in Inv(\overline{q}(i))$ for all $t \in [\tau_i, \tau_i')$, if $i \in \langle\tau\rangle$,and $y^i(\tau_i') \in Inv(\overline{q}(i))$, if $i = N(\tau)$ i $\tau_i' \in U(\tau)$;

2) $(\overline{q}(i+1), y^{i+1}(\tau_{i+1})) \in Jump(\overline{q}(i), y^i(\tau_i'), x)$ for all $i \in \langle\tau\rangle \setminus \{N(\tau)\}$;

3) function y^i is locally absolutely continuous on I_i and satisfies the equation $\dot{y}^i(t) = g(\overline{q}(i), y^i(t)) + h(\overline{q}(i), y^i(t))\dot{w}(t,x)$ for almost all $t \in I_i$;

4) $(q(0), y^0(0)) \in Init$.

Note that in clause 3 functions $t' \mapsto g_{\overline{q}(i)}(y^i(t'))$ i $t' \mapsto h_{\overline{q}(i)}(y^i(t'))$ are determined on $I_i \setminus \{\tau_i'\}$ because $y^i(t) \in Inv(\overline{q}(i))$ for all $t \in [\tau_i, \tau_i')$ in clause 1.

Consider fixed FHAFS $HA = (Q, Y, PS, g, w, h, Inv, Init, Jump)$.

Let $Orb(HA)$ be a set of phase orbit of automaton HA, φ_w – distribution function of a process fuzzy walk w.

Definition 3. Stationary state $y_* \in Y$ is a point that

1) $Jump(q, y_*, x) \subseteq Q \times \{y_*\}$ for all $q \in Q$ i $x \in X_+$;

2) for each $\chi = (\tau, \bar{q}, \bar{y}, x) \in Orb$ that $y^0(\tau_0) = y_*$, all maps y^i, $i \in \langle \tau \rangle$ are constant and take value y_*. Let $St(HA)$ be set of stationary states HA.

Definition 4. Stationary state $y_* \in St(HA)$ is stable with level of $\bar{\alpha}$, where $\bar{\alpha} : (0,+\infty) \to [0,1)$ is a function defined in a neighborhood of zero, if for any number $\varepsilon > 0$ exists a number $\delta > 0$, such as for all phase orbits $\chi = (\tau, \bar{q}, \bar{y}, x) \in Orb(HA)$, where $\bar{y} = (y^i)_{i \in \langle \tau \rangle}$ and $\tau = (I_i)_{i \in \langle \tau \rangle}$ such as $\left\| y^0(\tau_0) - y_* \right\| < \delta$ and $P\{x\} > \bar{\alpha}(\varepsilon)$, holds $\left\| y^i(t) - y_* \right\| < \varepsilon$ for all $i \in \langle \tau \rangle$ i $t \in I_i$.

Let $St(HA)$ be the set of *HA*'s stationary states.

Let $DF(\mathbb{R}^d)$ be the class of all partly defined continuous functions $f : \mathbb{R}^d \to \mathbb{R}^+$, which are continuously differentiable inside of their domain.

For each partly defined function $f : \mathbb{R} \to \mathbb{R}$ let's denote $Infinv(f)$ a partly defined function $g : \mathbb{R} \to \mathbb{R}$ fulfilling the conditions

• $g(y) = \inf\{x \in \mathbb{R} \mid f(x) \le y\}$, if $y \in \mathbb{R}$ and a set $\{x \in \mathbb{R} \mid f(x) \le y\}$ is not empty and upper-bounded;

• otherwise $g(y)$ is not determined.

Let $HLo(HA, y_*)$ be a set of tuples $\left(\bar{\alpha}, (V_q)_{q \in Q}, (v_q)_{q \in Q}\right)$ such as

– $\bar{\alpha} : (0,+\infty) \to [0,1)$ is a function defined in a neighborhood of zero,

– $(V_q)_{q \in Q}$ is a family of functions indexed class $DF(\mathbb{R}^d)$,

– $(v_q)_{q \in Q}$ is an indexed family of predefined functions $v_q : \mathbb{R}^+ \to \mathbb{R}^+$, defined in a neighborhood of zero (including zero), and the conditions in which $\psi = Infinv(\bar{\alpha})$ hold:

Lo1) $\{y_*\} \cup O_q \subseteq DomV_q$ for all $q \in Q$, where O_q is some open superset of $Inv(q)$, and if $(q_2, y_2) \in Jump(q_1, y_1, x)$ for some $q_1, q_2 \in Q$, $y_1, y_2 \in Y$ and $x \in X_+$, then $y_1 \in DomV_{q_1}$ i $y_2 \in DomV_{q_2}$;

Lo2) $v_q(0) = 0$ i $v_q(w) > 0$ for all $w \in Domv_q \setminus \{0\}$;

Lo3) $V_q(y_*) = 0$ for all $q \in Q$;

Lo4) if $V_q(y) \le v_q(w)$ for some $y \in DomV_q$ and $w \in Domv_q$, then $\left\| y_* - y \right\| \le w$;

Lo5) for all elements $q \in Q$, $x \in X_+$ and $y \in Inv(q)$ such that $P\{x\} \in Dom\psi$ and $V_q(y_1) > \nu_q(\psi(P\{x\}))$, the following inequality is defined and holds:

$$\nabla_{g_q} V_q(y) \le -\frac{1}{\kappa_w} \sqrt{\phi_w^{-1}(P\{x\})} \left\| \nabla_{h_q} V_q(y) \right\|.$$

We introduce the following notation (where $q_1, q_2 \in Q, u \in (0,1]$):

$$J(q_1, q_2, u) = \{(y_1, y_2) \mid \exists x \in X : P\{x\} >$$
$$> u \wedge (q_2, y_2) \in Jump(q_1, y_1, x)\}$$
$$E = \{(q_1, q_2) \in Q \times Q \mid \exists x \in X : P\{x\} > 0 \wedge$$
$$\wedge \exists (y_1, y_2) : (q_2, y_2) \in Jump(q_1, y_1, x)\}$$

Theorem 1. (cyclic stability of stationary states) Let HA be a FHAFS that switches states in cycle $\hat{q}_1 \to \hat{q}_2 \to \ldots \to \hat{q}_n \to \hat{q}_1$. Suppose that for stationary state $y_* \in St(HA)$ a tuple exists

$$HL = \left(\bar{\alpha}, (V_q)_{q \in Q}, (\nu_q)_{q \in Q}\right) \in HLo(HA, y_*).$$

Suppose the following conditions:

1) for each edge $(q_1, q_2) \in E$ there isa number $\delta_{q_1 q_2} > 0$ and map $\vartheta_{q_1 q_2} : [0, \delta_{q_1 q_2}] \to \mathbb{R}^+$ such that $\vartheta_{q_1 q_2}(0+) = \vartheta_{q_1 q_2}(0) = 0$ for all elements $u \in D$ and pairs $(y_1, y_2) \in J(q_1, q_2, u)$ inequality holds:

– $V_{q_2}(y_2) \le v_{q_2}(\psi(u))$, if $V_{q_1}(y_1) \le v_{q_1}(\psi(u))$,

– $V_{q_2}(y_2) \le \vartheta_{q_1 q_2}(V_{q_1}(y_1))$, if $V_{q_1}(y_1) \in [0, \delta_{q_1 q_2}]$ i $V_{q_1}(y_1) > v_{q_1}(\psi(u))$;

2) $\lambda(\hat{q}_1 \hat{q}_2 .. \hat{q}_{n-1} \hat{q}_n \hat{q}_1) \le_{SD} \lambda(\hat{q}_1)$, i.e. there is s' condition.

Thuss tationary state y_* of automaton HA is stable with a level of $\overline{\alpha}$.

Pulsed fuzzy hybrid automata

We prove a theorem on stability of stationary states of cyclic pulsedFHAFS.

Theorem 2. Let automaton *HA* be pulse, $(V_q)_{q \in Q}$ is a family of functions of class $DF_0^{\infty}(\mathrm{R}^d, y_*)$, such, that $O_q \subseteq \mathrm{Dom}\, V_q$ for each $q \in Q$, where O_q is some open superset of $Inv(q)$, and for all $q_1, q_2 \in Q$, $y_1, y_2 \in Y$, $x \in X_+$, such that $(q_2, y_2) \in Jump(q_1, y_1, x)$ holds $y_1 \in DomV_{q_1}$ and $y_2 \in DomV_{q_2}$. Let $(v_q)_{q \in Q}$ be a family of maps $v_q : \mathrm{R}^+ \to \mathrm{R}^+$ such that $v_q(0) = 0$ and $0 < v_q(w) < V_q(y)$ for all $w > 0$ and $y \in DomV_q$ such that $\|y_* - y\| > w$.

Suppose that for all $q \in Q$, $x \in X_+$ i $y \in Inv(q)$ such that $P\{x\} \in Dom\psi$ and $V_q(y_1) > \nu_q(\psi(P\{x\}))$ the inequality holds

$$\nabla_{g_q} V_q(y) \le -\frac{1}{\kappa_w}\sqrt{\phi_w^{-1}(P\{x\})}\left\|\nabla_{h_q} V_q(y)\right\|.$$

Then let's suppose that for all edges $(q_1, q_2) \in E$ a number $\delta_{q_1 q_2} > 0$ and a map $\vartheta_{q_1 q_2} : [0, \delta_{q_1 q_2}] \to \mathrm{R}^+$ exist such that $\vartheta_{q_1 q_2}(0+) = \vartheta_{q_1 q_2}(0) = 0$ and for all elements $u \in D$ and pairs $(y_1, y_2) \in J(q_1, q_2, u)$ inequalities hold

a) $V_{q_2}(y_2) \le v_{q_2}(\psi(u))$, if $V_{q_1}(y_1) \le v_{q_1}(\psi(u))$,

b) $V_{q_2}(y_2) \le \vartheta_{q_1 q_2}(V_{q_1}(y_1))$, if $V_{q_1}(y_1) \in [0, \delta_{q_1 q_2}]$ and $V_{q_1}(y_1) > v_{q_1}(\psi(u))$.

Suppose that values $d_i(u,v) \in \mathrm{R}^+, i = \overline{1,n}$ by all $(u,v) \in D \times [0, v_{\max}]$ and inequality $d_n(u,v) \le \max\{\nu_{\hat{q}_1}(\psi(u)), v\}$ for all $(u,v) \in D \times [0, v_{\max}]$, where

$$d_1(u,v) = \sup\left\{\begin{matrix} V_{\hat{q}_2}(y_2) \mid (y_1, y_2) \in J(\hat{q}_1, \hat{q}_2, u) \wedge \\ \wedge V_{\hat{q}_1}(y_1) \le \max\{\nu_{\hat{q}_1}(\psi(u)), v\} \end{matrix}\right\};$$

$$d_2(u,v) = \sup\left\{\begin{matrix} V_{\hat{q}_3}(y_3) \mid (y_2, y_3) \in J(\hat{q}_2, \hat{q}_3, u) \wedge \\ \wedge V_{\hat{q}_2}(y_2) \le \max\{\nu_{\hat{q}_2}(\psi(u)), d_1(u,v)\} \end{matrix}\right\};$$

$$d_3(u,v) = \sup\left\{\begin{matrix} V_{\hat{q}_4}(y_4) \mid (y_3, y_4) \in J(\hat{q}_3, \hat{q}_4, u) \wedge \\ \wedge V_{\hat{q}_3}(y_3) \le \max\{\nu_{\hat{q}_3}(\psi(u)), d_2(u,v)\} \end{matrix}\right\};$$

$$d_n(u,v) = \sup\left\{\begin{matrix} V_{\hat{q}_1}(y_1) \mid (y_n, y_1) \in J(\hat{q}_n, \hat{q}_1, u) \wedge \\ V_{\hat{q}_n}(y_n) \le \max\{\nu_{\hat{q}_n}(\psi(u)), d_{n-1}(u,v)\} \end{matrix}\right\}.$$

Then the stationary state y_* of pulsed cycle FHAFS HA is stable with level $\overline{\alpha}$.

Proof. We verify the conditions of theorem 1. Prove that $\left(\overline{\alpha}, (V_q)_{q \in Q}, (\nu_q)_{q \in Q}\right) \in HLo(HA, y_*)$. The condition of the theorem implies that the conditions Lo1, Lo3, Lo5 run. Conditions Lo2 performed as $v_q(w) > 0$ at $w > 0$ i $v_q(0) = 0$. Condition Lo4 performed, by the theorem if $\|y_* - y\| > w$, то $V_q(y) > v_q(w)$.

The condition of the theorem implies the condition 1 of Theorem 1.

Verify the condition 2of Theorem 1. As mentioned $d_i(u,v) \in \mathrm{R}^+, i = \overline{1,n}$ for all $(u,v) \in D \times [0, v_{\max}]$, then for all edges (q_1, q_2) we have

$$(\overline{c}(q_1) \cdot \overline{c}(q_1, q_2))(u,v) =$$
$$\sup s((q_1, q_2), u, \overline{c}(q_1)(u,v)) = d_i(u,v)$$

where s is a map from definition of λ markup. Then $d_n(u,v)=(\overline{c}(\hat{q}_1)\cdot\overline{c}(\hat{q}_1,\hat{q}_2)\cdot\overline{c}(\hat{q}_2)\cdot\overline{c}(\hat{q}_2,\hat{q}_3)\cdot\ldots\cdot \overline{c}(\hat{q}_n)\cdot\overline{c}(\hat{q}_n,\hat{q}_1))(u,v)\le\max\{\nu_{\hat{q}_1}(\psi(u)),v\}$ for all $(u,v)\in D\times[0,v_{\max}]$. Then $\max\{\nu_{\hat{q}_1}(\psi(u)),d_n(u,v)\}\le\max\{\nu_{\hat{q}_1}(\psi(u)),v\}$, and $\overline{c}(\hat{q}_1)(u,d_n(u,v))\le\overline{c}(\hat{q}_1)(u,v)$ where for all $(u,v)\in D\times[0,v_{\max}]$ holds

$$\lambda(\hat{q}_1\hat{q}_2..\hat{q}_{n-1}\hat{q}_n\hat{q}_1)(u,v)\le\lambda(\hat{q}_1)(u,v).$$

Then by definition of $\le_{SD}$ (for fixeda $v_{\max}$ and set D) holds

$$\lambda(\hat{q}_1\hat{q}_2..\hat{q}_{n-1}\hat{q}_n\hat{q}_1)\le_{SD}\lambda(\hat{q}_1).$$

Thus the conditions of the theorem (1) hold, and a stationary state y_* of automaton HA is stable with level $\overline{\alpha}$, that concludes the proof.

Let $d'\le d$ be a natural number, where d is the dimension of space of continuous states of HA. Let $L\in\mathbb{R}^{d'\times d}$ be a matrix of rank d'. Denote ρ_L pseudometrics on $\mathbb{R}^d$ defined by equality $\rho_L(x,y)=(Lx-Ly)^T(Lx-Ly)$.

Definition 5. Stationary state $y_*\in St(HA)$ is an L-stability with level $\overline{\alpha}$, where $\overline{\alpha}:(0,+\infty)\to[0,1)$ is a function defined in a neighborhood of zero, if for any number $\varepsilon>0$ exists a number $\delta>0$ such that for all phase orbits $\chi=(\tau,\overline{q},\overline{y},x)\in Orb(HA)$, where $\overline{y}=(y^i)_{i\in\langle\tau\rangle}$ and $\tau=(I_i)_{i\in\langle\tau\rangle}$, such that $\left\|y^0(\tau_0)-y_*\right\|\le\delta$ і $P\{x\}>\overline{\alpha}(\varepsilon)$ holds $\rho_L(y^i(t),y_*)\le\varepsilon$ for all $i\in\langle\tau\rangle$ and $t\in I_i$.

Theorem 3. Let HA be a FHAFS with cyclic discrete switching states $\hat{q}_1\to\hat{q}_2\to\ldots\to\hat{q}_n\to\hat{q}_1$. For stationary state $y_*\in St(HA)$ exists a tuple

$$HL=\left(\overline{\alpha},(V_q)_{q\in Q},(\nu_q)_{q\in Q}\right)\in HLo^L(HA,y_*).$$

Suppose the following conditions:

1) for each switching $q_1\to q_2$ there is a number $\delta_{q_1q_2}>0$ and a map $\vartheta_{q_1q_2}:[0,\delta_{q_1q_2}]\to\mathbb{R}^+$ such that $\vartheta_{q_1q_2}(0+)=\vartheta_{q_1q_2}(0)=0$ and for all elements $u\in Dom\psi$ and pairs $(y_1,y_2)\in J(q_1,q_2,u)$ inequalities hold

– $V_{q_2}(y_2)\le v_{q_2}(\psi(u))$, if $V_{q_1}(y_1)\le v_{q_1}(\psi(u))$,

– $V_{q_2}(y_2)\le\vartheta_{q_1q_2}(V_{q_1}(y_1))$, if $V_{q_1}(y_1)\in[0,\delta_{q_1q_2}]$ and $V_{q_1}(y_1)>v_{q_1}(\psi(u))$;

2) $\lambda(\hat{q}_1\hat{q}_2..\hat{q}_{n-1}\hat{q}_n\hat{q}_1)\le_{SD}\lambda(\hat{q}_1)$, i.e. s' condition.

Then stationary state y_* of automaton HA is L-stable with level $\overline{\alpha}$.

Let $HLo^L(HA,y_*)$ be a set of tuples $\left(\overline{\alpha},(V_q)_{q\in Q},(\nu_q)_{q\in Q}\right)$, in which:

– $\overline{\alpha}:(0,+\infty)\to[0,1)$ is a function defined in a neighborhood of zero;

– $(V_q)_{q\in Q}$ is an indexed family of functions of class $DF(\mathbb{R}^d)$;

– $(\nu_q)_{q\in Q}$ is an indexed family of functions $v_q:\mathbb{R}^+\to\mathbb{R}^+$, defined in a neighborhood of zero (including zero).

Let conditions Lo1, Lo2, Lo3, Lo5 hold, and, in addition, such a condition (instead Lo4):

Lo4') if $V_q(y)\le v_q(w)$ for some $y\in\mathrm{Dom}\,V_q$ and $w\in\mathrm{Dom}\,v_q$, then $\left\|Ly_*-Ly\right\|\le w.$

Theorem 4. (about L-stable steady states not cyclical pulse FHAFS). Let automata HA not pulse, $(V_q)_{q\in Q}$ – family of functions of class $DF_0^\infty(\mathbb{R}^d,y_*)$, such that $O_q\subseteq\mathrm{Dom}V_q$ for each $q\in Q$, де O_q – some open superset $Inv(q)$, and for all $q_1,q_2\in Q$, $y_1,y_2\in Y$, $x\in X_+$, such that $(q_2,y_2)\in Jump(q_1,y_1,x)$, performed $y_1\in\mathrm{Dom}V_{q_1}$ and $y_2\in\mathrm{Dom}V_{q_2}$.

Let $(\nu_q)_{q\in Q}$ be a family of maps $v_q:\mathbb{R}^+\to\mathbb{R}^+$ such as $v_q(0)=0$ and $0<v_q(w)<V_q(y)$ for all $w>0$ і $y\in\mathrm{Dom}\,V_q$, such that $\left\|Ly_*-Ly\right\|>w$.

Suppose that for all $q\in Q$, $x\in X_+$ and $y\in Inv(q)$, such that $P\{x\}\in\mathrm{Dom}\,\psi$ and $V_q(y_1)>\nu_q(\psi(P\{x\}))$ inequality holds

$$\nabla_{g_q}V_q(y)\le-\frac{1}{\kappa_w}\sqrt{\phi_w^{-1}(P\{x\})}\left\|\nabla_{h_q}V_q(y)\right\|.$$

Suppose that for all edges $(q_1,q_2)\in E$, $u\in D$ and $(y_1,y_2)\in J(q_1,q_2,u)$, such that $V_{q_1}(y_1)\le v_{q_1}(\psi(u))$, performed $V_{q_2}(y_2)\le v_{q_2}(\psi(u))$. Suppose also that values $d_i(u,v)\in \mathrm{R}^+, i=\overline{1,n}$ defined for all $(u,v)\in D\times[0,v_{\max}]$ and inequality $d_n(u,v)\le \max\{\nu_{\hat q_1}(\psi(u)),v\}$ hold for all $(u,v)\in D\times[0,v_{\max}]$, where

$$d_1(u,v)=\sup\begin{Bmatrix}V_{\hat q_2}(y_2)\,|\,(y_1,y_2)\in J(\hat q_1,\hat q_2,u)\wedge\\ V_{\hat q_1}(y_1)\le \max\{v_{\hat q_1}(\psi(u)),v\}\end{Bmatrix};\ d_2(u,v)=\sup\begin{Bmatrix}V_{\hat q_3}(y_3)\,|\,(y_2,y_3)\in J(\hat q_2,\hat q_3,u)\wedge\\ \wedge V_{\hat q_2}(y_2)\le \max\{v_{\hat q_2}(\psi(u)),d_1(u,v)\}\end{Bmatrix};$$

$$d_3(u,v)=\sup\begin{Bmatrix}V_{\hat q_4}(y_4)\,|\,(y_3,y_4)\in J(\hat q_3,\hat q_4,u)\wedge\\ \wedge V_{\hat q_3}(y_3)\le \max\{v_{\hat q_3}(\psi(u)),d_2(u,v)\}\end{Bmatrix};\ .....$$

$$d_n(u,v)=\sup\begin{Bmatrix}V_{\hat q_1}(y_1)\,|\,(y_n,y_1)\in J(\hat q_n,\hat q_1,u)\wedge\\ \wedge V_{\hat q_n}(y_n)\le \max\{v_{\hat q_n}(\psi(u)),d_{n-1}(u,v)\}\end{Bmatrix}.$$

Thus steady state $y*$ *not pulse cycle* FHAFS HA is L -stable with level $\overline{\alpha}$.

Proof. We verify the conditions of the theorem (3). Prove that $\left(\overline{\alpha},(V_q)_{q\in Q},(\nu_q)_{q\in Q}\right)\in HLo^L(HA,y_*)$. The condition of the theorem implies that the conditions Lo1, Lo3, Lo5 run. Conditions Lo2 performed as $v_q(w)>0$ at $w>0$ і $v_q(0)=0$. Condition Lo4' executed because by theorem if $\|Ly_*-Ly\|>w$, то $V_q(y)>\nu_q(w)$.

The condition of the theorem implies implementation of the first inequality of condition 1 Theorem (3).
Verify the implementation of the other inequality of condition 1of theorem (3). Granted $\delta_{q_1q_2}=\nu_{q_1}(0)>0$ and define the mapping $\vartheta_{q_1q_2}:[0,\delta_{q_1q_2}]\to \mathrm{R}^+$ equality

$$\vartheta_{q_1q_2}(v)=\sup\{V_{q_2}(y)\,|\,y\in \mathrm{Dom}\,V_{q_1}\cap$$
$$\cap \mathrm{Dom}\,V_{q_2}\wedge V_{q_1}(y)\le v\}.$$

Note that this value is determined for $y_*\in \mathrm{Dom}\,V_{q_1}\cap \mathrm{Dom}\,V_{q_2}$ and finite, and function V_{q_2} is locally limited, and set $V_{q_1}(y)\le v$ is limited (для $v>0$) because V_{q_1} allows unlimited extension. Because $V_{q_1}(y)>0$ at $y\ne y_*$, то $\vartheta_{q_1q_2}(0)=0$. Because $\vartheta_{q_1q_2}$ is monotonous, $\vartheta_{q_1q_2}(0+)$ is defined. Verify that $\vartheta_{q_1q_2}(0+)=0$. Really $\inf_{\varepsilon>0}\vartheta_{q_1q_2}(\nu_{q_1}(\varepsilon))=0$, because function V_{q_1} is continuous on its domain and $V_{q_1}(y_*)=0$. But if $\vartheta_{q_1q_2}(0+)\ne 0$, then $\vartheta_{q_1q_2}(0+)>0$ and $\inf_{\varepsilon>0}\vartheta_{q_1q_2}(\nu_{q_1}(\varepsilon))>0$, because $\nu_{q_1}(\varepsilon)>0$ at $\varepsilon>0$. Thus $\vartheta_{q_1q_2}(0+)=0$.

If $(y_1,y_2)\in J(q_1,q_2,u)$ for some $u\in D$ і $V_{q_1}(y_1)\le\delta_{q_1q_2}$, then $y_1=y_2$, and automata is not pulsed for $y_1\in \mathrm{Dom}\,V_{q_1}\cap \mathrm{Dom}\,V_{q_2}$. Then $V_{q_2}(y_2)\le\vartheta_{q_1q_2}(V_{q_1}(y_1))$ by definition $\vartheta_{q_1q_2}$.

Verify condition 2of theorem (3). As mentioned $d_i(u,v)\in \mathrm{R}^+, i=\overline{1,n}$ defined for all $(u,v)\in D\times[0,v_{\max}]$, then for all arcs (q_1,q_2),

$$(\overline{c}(q_1)\cdot\overline{c}(q_1,q_2))(u,v)=\sup s((q_1,q_2),u,\overline{c}(q_1)(u,v))=d_i(u,v),$$

where s isan image from definition of markup λ. Then

$$d_n(u,v)=(\overline{c}(\hat q_1)\cdot\overline{c}(\hat q_1,\hat q_2)\cdot\overline{c}(\hat q_2)\cdot\overline{c}(\hat q_2,\hat q_3)\cdot\ldots\cdot$$
$$\overline{c}(\hat q_n)\cdot\overline{c}(\hat q_n,\hat q_1))(u,v)\le \max\{\nu_{\hat q_1}(\psi(u)),v\}$$

For all $(u,v)\in D\times[0,v_{\max}]$. Thus $\max\{\nu_{\hat q_1}(\psi(u)),d_n(u,v)\}\le \max\{\nu_{\hat q_1}(\psi(u)),v\}$, and then

$$\overline{c}(\hat q_1)(u,d_n(u,v))\le\overline{c}(\hat q_1)(u,v)$$

For all $(u,v)\in D\times[0,v_{\max}]$ performed

$$\lambda(\hat q_1\hat q_2..\hat q_{n-1}\hat q_n\hat q_1)(u,v)\le\lambda(\hat q_1)(u,v).$$

Then by definition $\le_{SD}$ (for fixed a $v_{\max}$ and set D) holds

$$\lambda(\hat q_1\hat q_2..\hat q_{n-1}\hat q_n\hat q_1)\le_{SD}\lambda(\hat q_1).$$

Thus conditions of (theorem 3) are satisfied, then steady state y_* automata HA є L -stable with level $\overline{\alpha}$.

The theorem is proved.

Conclusions

The paper examines properties of fuzzy hybrid autommata. For modeling of fuzziness we used approach based on the theory of possibilities. The questions of stability on a part of variables (L-stability) of solutions of fuzzy hybrid automata with cyclic changes in local conditions are mentioned. Change happens when states achieved a certaint rajectory set. The corresponding theorems are proved.

References:

1. Branicky M. Stability of switched and hybrid systems / M. Branicky // In Proc. 33-rd Conf. Decision and Control, LakeBuena Vista, FL, Dec. 1994. – P. 3498–3503.
2. Daafouz J. Stability Analysis and Control Synthesis for Switched Systems: A switched Lyapunov function approach / J. Daafouz, P. Riedinger, Candlung // IEEE transactions on automatic control. – 2002. – V. 47, N 1. – P. 1883–1887.
3. Lin H. Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results / H. Lin, P. J. Antsaklis // IEEE Transactions on Automatic Control. – 2009. – V. 54, N 2. – P. 308–322.
4. Stability analysis of singularly perturbed switched and impulsive linear systems / J. B. Rejeb, I. C. Morărescu, A. Girard, J. Daafouz // 2016 IEEE 55th Conference on Decision and Control (CDC), Las Vegas, NV, 2016. – P. 5521–5526.
5. Yang Li. Piecewise quadratic stability analysis combined with time information for hybrid system / Li Yang, H. Zhang // 2016 IEEE Advanced Information Management, Communicates, Electronic and Automation Control Conference (IMCEC), Xi'an, 2016. – P. 1905–1909.
6. Stability Analysis of Switched Linear Systems defined by Regular Languages / Y. Wang, N. Roohi, G. E. Dullerud, M. Viswanathan // IEEE Transactions on Automatic Control. – Vol. PP, No. 99. – P. 1.
7. Бичков О. Дослідження стійкості тривіальних фазових орбіт гібридних автоматів / О. Бичков // Вісн. Київ. ун-ту. Серія "Кібернетика". – 2005. – № 6. – С. 4–8.
8. Бичков О. С. Дослідження стійкості імпульсних гібридних автоматів / О. С. Бичков // Математические машины и системы. – 2007. – № 1. – С. 27–33.
9. Бичков О. С. Побудова (PN)–моделі теорії можливостей / О. С. Бичков, К. С. Колесніков // Вісн. Київ. ун-ту. Сер. "Фіз.-мат. науки". – 2007. – № 1. – С. 134–138.
10. Бичков О. С. До теорії можливостей та її застосування / О. С. Бичков // Доповіді НАН України. – 2007. – № 5. – С. 7–12.

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О. С Бичков, канд. фіз.-мат. наук, доц.
Київський національний університет імені Тараса Шевченка, Київ, Україна

МОДЕЛІ Й МЕТОДИ ДОСЛІДЖЕННЯ ІМПУЛЬСНИХ НЕПЕРЕРВНО-ДИСКРЕТНИХ ІНФОРМАЦІЙНИХ СИСТЕМ І ПРОЦЕСІВ

Досліджено стійкість нового класу неперервно-дискретних інформаційних систем – нечітких лінійних гібридних імпульсних автоматів. Отримано достатні умови стійкості. Доведено стійкість і стійкість за частиною змінних. Описано циклічний випадок. Дослідження виконувалися методом функцій Ляпунова.

Ключові слова: дискретно-неперервні системи, стійкість, метод функцій Ляпунова.

А. С. Бычков, канд. физ.-мат. наук, доц.
Киевський национальний университет имени Тараса Шевченка, Киев, Украина

МОДЕЛИ И МЕТОДЫ ИССЛЕДОВАНИЯ ИМПУЛЬСНЫХ НЕПРЕРЫВНО-ДИСКРЕТНЫХ ИНФОРМАЦИОННЫХ СИСТЕМ И ПРОЦЕССОВ

Рассмотрена устойчивость нового класса непрерывно-дискретных информационных систем – нечетких линейных гибридных импульсных автоматов. Получены достаточные условия устойчивости. Доказана устойчивость и устойчивость по части переменных. Описаны циклические случаи. Исследования проводились методом функций Ляпунова.

Ключевые слова: дискретно-непрерывные системы, устойчивость, метод функций Ляпунова.

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N. I. Garkusha, PhD, senior researcher
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

EVALUATION OF PROCESSES DYNAMICS IN ARTIFICIAL INTELLIGENCE NEURODYNAMICS PROBLEMS

The paper considers dynamic processes in the problems of artificial intelligence, namely, in problems of neurodynamics. The stability problems of dynamical processes and estimates of their convergence are examined. In particular, the mathematical model of neural network dynamics proposed by Hopfield is considered. Conditions for the convergence of the process are formulated in terms of the second Lyapunov method.

Keywords: neurodynamics, stability, Lyapunov's second method, Hopfield model.

Introduction. Today, the problems of creating new effective information technologies for analyzing and processing large information flows come to the forefront before the scientific community. Mankind has faced a paradoxical situation, that with the growth of information, there is a decrease in awareness, which complicates the analytical activities in various fields. This calls for the emergence of new areas of science and technology, one of which is artificial intelligence [1-3]. The birth of artificial intelligence is attributed to 1956 after the organization of a two-month seminar at the Dortmund College. This seminar allowed us to declare artificial intelligence as the birth of a new scientific direction. The first works appeared after 1945, and the name "artificial intelligence" itself was proposed in 1956 [2]. The main directions of development were identified in the fundamental publications [2,3]. It was noted that artificial intelligence is a scientific direction, the main tasks of which are [3, p.35]:

1. developing systems that think like people;
2. developing systems that think rationally.

The prehistory of artificial intelligence is based on such sciences as philosophy, mathematics, economics, psychology, medicine, computer science, management theory and cybernetics, linguistics. As noted in [4, pp. 18-19], in order to construct a solution to a problem, it is necessary, first, to formalize the procedure of its formulation, and second, the procedure of finding its solution. To do this, preliminary, the problem itself must be represented either in the framework of an approach based on the state space or as a theorem requiring proof. The procedure for finding a solution can be either "instantaneous" or "iterative", i.e. multistep. The second procedure entails the construction of an algorithm for "improving" the representation of the solution of problems based on an assessment of the adequacy of the resulting intermediate solutions to the true one. It requires the installation of a "criterion for the accuracy of the problem's solution" or "perturbation evaluation". In addition, you can establish additional evaluation characteristics. The simplest apparatus for finding a solution can be a system of differential equations.

One of the fundamental directions of the development of artificial intelligence is the modeling of processes that occur in the human brain. Since most processes occur during a certain period of time, i.e. are dynamic, then the methods of constructing and investigating mathematical models of dynamical systems are of particular importance.

The proposed article considers the direction of artificial intelligence, which is associated with dynamic processes in neural networks, the so-called direction of neurodynamics. The dynamic process acquires the completeness property, if it is convergent, i.e. is asymptotically stable. One of the basic methods for investigating the stability of dynamic processes of a different nature is the second, or direct, Lyapunov method. The main results of its use in the dynamics of neural networks are given.

Dynamic processes in neural networks. The problems of artificial intelligence, like the study of the possibilities of modeling human thought processes, have been considered for a long time. In their solution, various approaches were proposed. One of the important approaches of artificial intelligence is the theory of neural networks [10-12]. Neural networks are a section related to building an apparatus similar to an artificial brain. The main directions of neural networks are training and the actual functioning of the apparatus, created by analogy with the human brain. If the processes of solving problems are not instantaneous, but iteratively, then the process is dynamic. In general, any direction related to the processes of learning and the system functioning occurs on the timeline, and therefore it can be considered a dynamic process.

The simplest mathematical apparatus used to describe dynamic processes is differential equations and systems of equations.

The construction of mathematical models describing the phenomena of learning and the functioning of artificial neural networks is part of the problem of artificial intelligence.

Dynamics of neural networks. ***One of the scientific directions of artificial intelligence, related to dynamics, is the simulation of processes in neural networks. In 1943, McCulloch and Walter Pitts [10] suggested that the nerve cells of the brain be regarded as logical elements, and the cell system assembled into the network as an elementary computing device capable of imitating logical elements. To those same scientists also belongs the primacy in the definition of "neural networks".***

The neural networks, in which the output signal is fed back to the input, an iterative process arises – a network with feedback is obtained. This network structure is called auto-associative. The described type was first proposed by Hopfil in 1982 [11]. The mathematical model of neural network dynamics could be described by a system of difference equations.

The model of a neuron, represented as an approximation of an electrical circuit in the form of a distributed transmission line of a biological dendritic neuron, is considered in [12]. This nature of the RC chain can be explained by the fact that the biological synapse itself is a filter designed for good approximation. In physical terms, synaptic weights w_{ij} , are the capacitances, and the corresponding output signals $x_i(t)$, $i=\overline{1,n}$ potentials, where n – number of inputs . These signals are fed to summing connections. The total current can be written in the form $\sum_{i=1}^{n} w_{ij}x_j(t)+I_i$, where the first term reflects the excitations acting on the synaptic weights, and the second term is the current source representing the external bias. Let $v_i(t)$ induced local field at the input of a nonlinear activation function $\varphi(\bullet)$. Then the total current flowing from the input node of the nonlinear element can be expressed in the form $\frac{v_i(t)}{R_i}+C_i\frac{dv_i(t)}{dt}$, where the first part is leakage resistance R_i , and the second on is capacitance C_i . Applying Kirchhoff's law, we obtain a system of equations

$$C_i\frac{dv_i(t)}{dt}=-\frac{v_i(t)}{R_i}+\sum_{j=1}^{n}\omega_{ij}x_j(t)+I_j .$$

For a given induced local field $v_i(t)$ you can determine the yield of a neuron i using relation $x_j(t)=\phi(v_j(t))$.

Activation function $\varphi(\bullet)$, determining output $x_j(t)$ neuron j to its induced local field $v_j(t)$ is a continuously differentiable function. Most often as an activation function a logical function $\varphi(v_j)=\frac{1}{1+\exp\{-v_j\}}$, $j=\overline{1,n}$ is used. Doing the replacement $a_i=1/R_iC_i$, $w_{ij}=\omega_{ij}/C_i$, we obtain the system of differential equations

$$\frac{dv_i(t)}{dt}=-a_iv_i(t)+\sum_{j=1}^{n}w_{ij\,j}\varphi(\nu_j(t))+K_j .$$

Another model of neurodynamics can be described by a system of differential equations

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \varphi\left(\sum_{j=1}^{n} w_{ij} x_j(t)\right).$$

Neurons connected to the network form a neural network. The Hopfield network consists of a set of neurons forming multiple-loop feedback system. The number of feedbacks is equal to the number of neurons. The output of each neuron is closed through an element of a single delay to all other network neurons. And the neuron of this network does not have feedbacks to itself. Equations of dynamics of the Hopfield model can be rewritten in the form

$$C_i \frac{dv_i(t)}{dt} = -\frac{v_i(t)}{R_i} + \sum_{j=1}^{n} \omega_{ij}\varphi_j(v_j(t)) + I_j .$$

Difference systems in neural networks. ***In addition to differential equations, dynamic processes in neural networks are adequately described by systems of difference equations [15]. The basic propositions are similar to those adopted for systems of differential equations.***

Problems of stability and convergence of processes of dynamics in neural networks. One of the important tasks of dynamic processes in neural networks is the study of convergence processes, which are provided by the conditions of asymptotic stability of the solutions of the system. In the theory of dynamical systems stability problems are solved on the basis of the second Lyapunov method. The essence of it is following. If there exists a positive-definite function whose total derivative is negative-definite by virtue of the system, then the equilibrium position is asymptotically stable. For difference systems, the condition of negative definiteness is replaced by the condition of negative definiteness of the first difference of solutions by virtue of the system. Usually, a positive definite function (the Lyapunov function) is chosen from physical assumptions and it is the total energy of the system. Нейронная сеть будет обучаема, если процесс является сходящимся. In work [12] the general principle of achievement of stability of a class of neural networks, described by the following system

$$\frac{d}{dt}u_j(t) = a_j(u_j(t))[b_j(u_j(t)) - \sum_{i=1}^{N} c_{ji}\varphi_i(u_i(t))],\ j = \overline{1, N} .$$

The Lyapunov function was constructed in the form

$$E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\varphi_i(u_i)\varphi_j(u_j) - \sum_{j=1}^{N}\int_0^{u_j} b_j(\lambda)\varphi_j'(\lambda)d\lambda,\ \varphi_j'(\lambda) = \frac{d}{d\lambda}\varphi_j(\lambda),\ j = \overline{1, N} .$$

Parameters of the system were superimposed on the conditions of symmetry, nonnegativity, and monotonicity. And the conditions of asymptotic stability, i.e. convergence of the dynamic process, were formulated in the Cohen-Grossberg theorem. Namely, if the system of nonlinear differential equations satisfies the conditions of symmetry, nonnegativity, and monotonicity, then the trivial solution of the system is globally stable.

Specific conditions for the convergence of dynamic processes in the models of neurodynamics for the system

$$\frac{dy_i(t)}{dt} = -a_i y_i(t) + \sum_{j=1}^{n}\omega_{ij}\varphi_j(y_j(t)) + I_i^0,\ t \ge 0,\ i = \overline{1, n}$$

were obtained in [14]. Let the system of equations

$$-a_i y_i + \sum_{j=1}^{n}\omega_{ij}\varphi_j(y_j) + I_i^0 = 0,\ i = \overline{1, n}$$

has a solution in the point $M_0(y_1^0, y_2^0, ..., y_n^0)$, $y_i^0 > 0$, $k = \overline{1, n}$. Making the substitution

$y_k(t) = x_k(t) + y_k^0$, $i = \overline{1, n}$, we obtain

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n}\omega_{ij}F_j(x_j(t)),\ i = \overline{1, n},\ F_j(x_j(t)) = \varphi_j(x_j(t) + y_j^0) - \varphi_j(y_j^0),\ j = \overline{1, n}.$$

After this substitution, the stability study of the equilibrium position $M_0(y_1^0, y_2^0, ..., y_n^0)$ of the initial system was reduced to the study of the stability of the zero equilibrium position of the transformed system. We introduce the following notation

$$c_{ii} = h_i(a_i - L_i\omega_{ii}),\ c_{ij} = -\frac{1}{2}(h_i L_j\omega_{ij} + h_j L_i\omega_{ji}),\ i, j = \overline{1, n},\ C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{12} & c_{22} & \cdots & c_{2n} \\ . & . & \cdots & . \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}.$$

The following conditions hold for asymptotic stability [14].

Theorem. Let the system of equations (6) have a solution $M_0\left(y_1^0, y_2^0, ..., y_n^0\right)$, $y_i^0 > 0$, $i = \overline{1,n}$, and constants exist $h_i > 0$, $i = \overline{1,n}$, under which a symmetric matrix C will be positively defined. Then the equilibrium $M_0\left(y_1^0, y_2^0, ..., y_n^0\right)$ is globally asymptotically stable.

To study stability, we use the Lyapunov function of quadratic form (kinetic energy)

$$V(x) = \frac{1}{2}\sum_{i=1}^{n} h_i x_i^2 \ . \ h_i > 0 \ , \ i = \overline{1,n} \ .$$

Stability of difference systems. The second Lyapunov method in difference systems. For difference systems, the concepts of stability and methods of their investigation are similar to those used for differential systems. We can consider the system of equations

$$y_i(k+1) = a_{ii} y_i(k) + \sum_{j=1}^{n} f_{ij}\left(y_j(k)\right) + I_i \ , \ i = \overline{1,n}$$

and obtain the stability conditions and asymptotic convergence. Moreover, the Lyapunov function also has the form of a quadratic form (the energy of the system), and the total derivative is replaced by the first difference by virtue of the system.

Rigidity of neural networks. In the study of nonlinear dynamical systems, the notion of "stability" needs to be clarified. In linear systems, equilibrium points are either points (equilibrium positions) or entire manifolds (hyperplanes). And all solutions are simultaneously either stable (asymptotically stable) or unstable depending on the stability (instability) of the zero equilibrium position (or manifold). For nonlinear systems this is not the case. In nonlinear systems, some equilibrium positions can be stable, others are not. There are stable (unstable) varieties and even quite complex structures ("strange attractors"). Neural networks have a huge number (millions) of neurons and the problem of stability here is much more complicated. Probably, it is more accurate to speak not about stability, but about the "roughness" of the dynamic system. The concept of a "rough system" appeared at the beginning of the last century when considering a system on a plane, i.e. two-dimensional system [13]. Necessary and sufficient conditions for the roughness of the nonlinear system were obtained. For three-dimensional systems, it was not possible to obtain similar results. For nonlinear dynamical systems described by differential equations in three-dimensional space, in the sixties of the last century a phenomenon called a "strange attractor" was discovered. Its essence lies in the fact that there is an asymptotically stable region ("pool of attraction"), within the trajectories "run up" and "behave unpredictably."

For large-scale systems (as a neural network appears), a study of the qualitative behavior of the system has not yet been carried out. Reduction of the multidimensional system of ordinary equations to the partial differential equation, which is carried out in problems of hydrodynamics, in this case does not make sense.

References:
1. Amosov N. M. Algorithms of mind / N. M. Amosov. – K. : Naukova dumka. – 85 p.
2. Russel S. Artificial intelligence / S. Russel, P. Norvig. – Second edition. – Moscow; St.-Petersburg; Kyiv, 2006. – 1407 p.
3. Luger G. F. Artifical intelligence. Strategies and methods of solutions of complex problems / G. F. Luger. – M. : Izdatelskiy dom "Viliyams", 2005. – 864 p.
4. Nilson N. Artificial intelligence / N. Nilson. – M. : Mir. – 1973. – 270 p.
5. Kostrov B. N. The basis of artificial intelligence / B. N. Kostrov, V. A. Ruckin, V. A. Fulin. – M. : DESS, TechBook, 2007. – 192 p.
6. Matveev M. G. Models and methods of artifical intelligence. Application in economics / M. G. Matveev, A. S. Sviridov, N. A. Aleinikova. – M. : Financy i statistika, Izdatelskiy dom "INFRA-M", 2008. – 448 p.
7. Yaschenko V. A. Artificial intelligence. Theory. Modelling. Application / V. A. Yaschenko. – K. : Logos, 2013. – 289 p.
8. Baydik T.N. Neuron networks and tasks of artificial intelligence / T. N. Baydik. – K : Naukova dumka, 2001. – 263 p.
9. Bondarev V. N. Artificial Intelligence / V. N. Bondarev, F. G. Ade. – Izd-vo SevNTU, 2002. – 615 p.
10. McCulloch W. S. A logical calculus of the ideas immanent in nervous activity / W. S. McCulloch, W. Pitts // Bulletin of Mathematica1 Biophysics. – 1943. – Vol. 5. – P. 115–133.
11. Hopfield J. J. Neurons with graded response have collective computational properties like those of two state neurons / J. J. Hopfield // Proceedings of the National Academy of Science. – USA, 1984. – Уol. 81. – P. 3088–3092.
12. Halkin S. C. Neural networks: full course / S. C. Halkin. – 2-nd edition. – M. : Izdatelskiy dom "Viliyams", 2006. – 1104 p.
13. Andronov A. A. Rough system / A. A. Andronov, L. S. Pontryagin // Reports AS URSS. – 1937. – T. 14, № 5.
14. Khusainov D. Ya. Estimates of the processes convergence in one model of Hopfield's neurodynamics / D. Ya. Khusainov, N. I. Garkusha // Bulletin of Taras Shevchenko National university of Kyiv. Series "Physics & Mathematics". – 2016. – V. 4 – P. 129–132.
15. Neuron systems in automatisation systems / V. I. Arhangelskiy, I. N. Bogaenko, G. G. Grabovskiy, N. A. Ryumshin. – K. : Technika, 1999. – 364 p.

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Н. І. Гаркуша, канд. екон. наук
Київський національний університет імені Тараса Шевченка, Київ, Україна

ОЦІНКИ ДИНАМІКИ ПРОЦЕСІВ У ЗАДАЧАХ НЕЙРОДИНАМІКИ ШТУЧНОГО ІНТЕЛЕКТУ

Розглянуто динамічні процеси в задачах штучного інтелекту, а саме в задачах нейродинаміки, проблеми стійкості динамічних процесів і оцінки їхньої збіжності, зокрема математичну модель динаміки нейронної мережі, запропоновану Хопфілдом. Умови збіжності процесу сформульовано в термінах другого методу Ляпунова.

Ключові слова: нейродинаміка, стійкість, другий метод Ляпунова, модель Хопфілда.

Н. И. Гаркуша, канд. экон. наук
Киевский национальный университет имени Тараса Шевченко, Киев, Украина

ОЦЕНКИ ДИНАМИКИ ПРОЦЕССОВ В ЗАДАЧАХ НЕЙРОДИНАМИКИ ИСКУССТВЕННОГО ИНТЕЛЛЕКТА

Рассмотрены динамические процессы в задачах искусственного интеллекта, а именно в задачах нейродинамики, проблемы устойчивости динамических процессов и оценки их сходимости, в частности математическая модель динамики нейронной сети, предложенная Хопфилдом. Условия сходимости процесса сформулированы в терминах второго метода Ляпунова.

Ключевые слова: нейродинамика, устойчивость, второй метод Ляпунова, модель Хопфилда.

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I. A. Dzhalladova, Dr. Sc. Physics and Mathematics, prof.,
V. K. Galicyn, Dr. Sc. Economic, prof.,
A. I. Babynuk, Ph.D. Economic, as.
Kyiv National Economics University named after Vadym Hetman, Kyiv, Ukraine

SYSTEM ANALYSIS OF CORRUPTION AS THREAT OF ECONOMIC SAFETY IN EU CYBERSPACE

The article is devoted to the issue of modeling corruption as a threat to the state's economic security, the study of the dependence of equilibrium levels of corruption on the model parameters, and focuses on the differences in the consequences of small and significant changes in parameters.

Keywords: corruption, dynamic model of corruption, economic security, equilibrium states.

Introduction.

Economic security is an integral part of national security, its foundation and material basis, the provision of which is the exclusive prerogative of the state. According to the "National Security Strategy of Ukraine", one of the topical threats to the national security of Ukraine is "the spread of corruption, its rooting in all spheres of state governance" (Section 3.3), "corrupt pressure on business" (paragraph 3.4), "criminalization and corruption energy sector "(item 3.5) [1].

Economic security involves providing conditions for maintaining economic activity and counteracting numerous attacks, among which the most important are: financial fraud; cybercrime; industrial espionage; corruption; shadow economy [2, c. 87]. According to a survey conducted by Ernst & Young auditors at the beginning of 2017, Ukraine was in the first place in terms of corruption among 41 countries in Europe, the Middle East, India and Africa (EMEIA), while in 2015 Ukraine ranked 7th in the list [3]

In the World Index of Corruption Perceptions for 2016, Ukraine gained 29 points out of 100 possible, which is 2 points more than last year. The improvement of the position in the world ranking of SRI promoted anti-corruption reform, but the lack of an effective judicial system and the actual impunity of corrupt officials do not allow Ukraine to make a powerful breakthrough and overcome the 30-ball barrier, called "a shame for the nation," said the experts of the organization [4]. The greatest challenges of the study were recognized by the weakness of institutions designed to ensure the rule of law, excessive regulation of the economy and the concentration of power in the hands of oligarchic clans. Business has seen significant corruption in the distribution of public funds, and the judiciary has found it incapable of damaging it.

The impact of corruption on the country's economic security as one of the most important components of national security is manifested, in particular, in the theft of state property and lobbying for private economic interests. If a state involved in an illegal corrupt business can not provide property security and legal business, the security deficit is filled in privately through the development of a security services market. The expenses incurred by economic entities in order to ensure the safety of their activities are covered by the increase in the value of goods and services. In addition, the spread of corruption in Ukraine is one of the main obstacles to the development of modern innovative technologies and the attraction of foreign investment in industry. In shadow corrupt schemes, a large amount of financial resources is turning, and the success of implementing large investment projects is often directly linked to the success of corruption. At the same time, illegal corrupt mechanisms are usually based on branched corruption schemes formed by groups of enterprising criminal officials who "solve problems" / "do not hinder legal activity" not only business structures but also state enterprises. Thus, as a result, final consumers are being paid for non-fulfillment of the duties of state authorities regarding the business security of the business. Or, in other words, citizens pay for corruption.

Historical Rewiew

The decisive role of the state is reflected in most definitions of corruption, which is conventionally understood and referred to as the private wealth-seeking behaviour of someone who represents the state and the public authority. It is the misuse of public goods by public officials, for private gains. The working definition of the World Bank is that corruption is the abuse of public power for private benefit [10]. Another widely used description is that corruption is a transaction between private and public sector actors through which collective goods are illegitimately converted into private regarding pay offs. Political or grand corruption takes place at the highest levels of political authority. It is when the politicians and political decision-makers, who are entitled to formulate, establish and implement the laws in the name of the people, are themselves corrupt, and use the political power they are armed with to sustain their power, status and wealth. It is when policy formulation and legislation is tailored to benefit politicians and legislators. Political corruption can thus be distinguished from bureaucratic or petty corruption, which is corruption in the public administration, at the implementation end of politics. Petty corruption has also been called low level and street level to name the kind of corruption that citizens will experience daily, at times, in their encounter with public administration and services like hospitals, schools, local licensing authorities, police, taxing authorities and so on [11]. The money or benefits collected through corruption is privatised to various degrees. It may be extracted for the benefit of an individual who will share nothing or very little with his equals, or it may be extracted for the benefit of a particular group with some coherence and unity. This has led to a second analytically important classification of corruption, namely between private and collective forms of corruption [12]. An understanding of corruption that include the possibility that it might be collective, institutionalised and organised in the interest of the ruling elite itself is vital to analyse the causes and logics of grand scale systemic corruption. To analyse who, in aggregate terms, benefits the most from corruption will also be required to understand the vested interests anti-corruption initiatives will be up against, and to identify the political/institutional levels most relevant for reform. The question who benefits from corruption could, however, be taken one step further. In aggregate terms corrupt practices will generate a flow of resources either from the society to the state (extractive corruption, or corruption from above), or from the state to the society (redistributive corruption, or corruption from below). In what can be called the theory of redistributive corruption, the state is the weaker part in the statesociety relationship. Analysts tend to have weak states in mind when they are focusing on the subversive groups and their state destructive capacity, i.e. the degenerative effects of corruption on the state institutions and the national economy.

Three paradigms for the analysis of corruption [14-16]. The **first** is the economic paradigm, which usually takes the principal-agent model of corruption as its founding pillar. In this paradigm corruption is considered the outcome of rational individual choices, and its spread within a certain organization is influenced by the factors defining the structure of expected costs and rewards. A **second approach** the cultural paradigm looks at the differences in cultural traditions, social norms and interiorized values which shape individualsmoral preferences and consideration of his social and institutional role. These are a leading forces that can push a corrupt public or private agent (not) to violate legal norms. A **third neo-institutional** approach considers also mechanisms which allow the internal regulation of social interactions within corrupt networks, and their effects on individuals beliefs and preferences. Though the corrupt agreements cannot be enforced with legal sanctions, several informal, non-written rules, contractual provisos and conventions may regulate the corrupt exchange between agent and corruptor. The A. underlines that corruption is the outcome of a multitude of individual and collective choices which change public opinion towards corruption and its diffusion throughout the state, markets and civil society. There is no univocal recipe to deal with anti-bribery measures, since corruption is a complex and multifaceted phenomenon. Reforms aimed at dismantling systemic corruption have to be finely tuned against its hidden governance structures, i.e. its internal regulation of exchanges and relationships. Otherwise, a vicious circle may emerge: the more an anti-corruption policy is needed, because corruption is systemic and enforced by effective third-parties, the less probable its formulation and implementation. Only when official rules are complemented by coherent informal institutions, bottomup initiatives, they tend to produce the expected outcomes and make anticorruption regulation more effective [17-20].

1. Problem statement.

State corruption is defined as the sale by state employees of state property for private purposes. Corruption is a general term that signifies the mercenary use of its position in society for personal purposes. This may be *bureaucratic* or *political* corruption. Corruption in the private market is usually not regulated by law, and a representative of one firm may bribe a representative of another, although the firms themselves have the right to dismiss the provincial employees.

Economic studies of the phenomenon of corruption as such actually began not later than 1975 with the work of Rose-Ackerman [5], in which corruption was seen as an economic behavior in terms of the risk associated with the commission of a crime and possible punishment for him.

There are many mathematical models of corruption in the literature at the present time. Conditionally, mathematical models of two directions can be distinguished – the study of acts of external corruption and corrupt organization from the inside. The third direction concerns the study of observed phenomena, such as the non uniformity of equilibrium corrupt states, the cyclicality of occurrence, etc.

2. Base analysis of corruption

1. If the goods differ and the costs are reduced when the contract is concluded, the goods are not sold in the private market, then firms have an incentive to bribe the bureaucrat. We will notice that in this case, all firms offer at the choice of the official goods, the same in terms of the ratio of **price — quality**, because all firms will try to put themselves in line with the dominant seller. Thus, the bureaucrat can choose each of the competitors, since any solution from the whole spectrum of **price — quality** has the same utility for the state. In this situation, firms can try to get a contract with a bribe. It is assumed that the bureaucrat will organize a market for bribes, telling truthfully to each firm about the largest of the already proposed.

Let G — the profit of the bureaucrat π_i – the profit of the seller i, then

$$G\left(X^i\right)=X^i-J\left(X^i\right)-R\left(X^i\right) \qquad (1)$$

$$\pi_i\left(X^i\right)=P^i\left(X^i\right)-T^i\left(X^i\right)-D^i\left(X^i\right)-N^i\left(X^i\right), \qquad (2)$$

where X^i – the size of the total bribe paid by the seller i, P^i – the price of the unit of the seller's product i; q – quantity of the product required by the state (provided by the given); $J\left(X^i\right)$ – the average fine for the bureaucrat. $J>0;$ $R\left(X^i\right)$ – moral costs for the bureaucrat when making a bribe X^i, in monetary terms, $R'\geq 0;$ T^i – total expenses for the production q of units for the seller i; $D^i\left(X^i\right)$ – average fine for the seller, $D'>0;$ $N^i\left(X^i\right)$ – moral costs for the seller when giving a bribe in monetary terms, $N'>0$.

The magnitude $J\left(X^i\right)$ that reflects the expected penalty for the bureaucrat can be determined by multiplying the average fine imposed in condemnation on the combined probability of arrest and condemnation. A similar procedure can be used to determine the expected penalty for the seller $D^i\left(X^i\right)$.

In terms of the profit of the bureaucrat and sellers, there is a set of bribes X^i that are permissible both for the bureaucrat and for the sales companies. In this area you can choose the optimal (in one or another content) the point (or value) of the bribe.

All bribes are acceptable to the bureaucrat. Four possible cases are considered $X\geq J(X)+R(X)$:

1. there are no acceptable bribes;
2. all bribes are acceptable, for example, as $J'+R'<1$ і $J(0)+R(0)=0$;
3. all bribes are acceptable to no more than a certain maximum level, since the marginal moral costs and / or the maximum expected fines increase with increasing X;
4. acceptable bribes that are smaller or equal to some minimum, because $\left(J_{xx}+R_{xx}\right)\leq 0$ and $J(0)+R(0)=0$.

The most plausible case 4, in which all bribes are taken, which are more or equal to a certain level $X_{\min}$.

The permissible area i of the seller is determined by the ratio $X^i \le P^i q - T^i - D^i\left(X^i\right) - N^i\left(X^i\right)$. So that a bribe is possible, a condition $P^i q - T^i > 0$ is required. This means that if not every corporation in the market is corrupt, then a potentially corrupt firm should earn excess profits – either because it works more efficiently than the extremely effective firm, or through the "entry barrier" to the market that brings the benefit to all vendors. For each seller i you can find the maximum possible bribe X_0^i that he is able to give. If $\max\left[X_0^i\right] = X_0^m$, then the firm m gets a contract. Such a rule, according to which the bureaucrat chooses a winning company.

Assuming that the expected fines for all firms will be the same, then the winning firm will be the one that will have the biggest difference between income and the amount of production and moral costs of bribes X_0^i. Because production and moral costs are treated in the same way, the size of the maximum bribe that the firm is willing to pay may fall – either because of increased costs of production, or because the representatives of the firm became more pedantic people.

2. In the situation with inaccurate formulations of the state, the model is complicated by introducing one more parameter Y^i : – the quality level. Increasing the price or reducing the quality of a product simply increases the probability of punishing the parties to the transaction. It is assumed that

$$J = J\left(P^i, Y^i, X^i\right), J_p \ge 0, J_y \le 0, J_x \ge 0, J\left(0, Y^i, X^i\right) = 0 \quad (3)$$

$$D = D\left(P^i, Y^i, X^i\right), D_p \ge 0, D_y \le 0, D_x \ge 0, D\left(0, Y^i, X^i\right) = 0 \quad (4)$$

Now it is possible that firms will want to give a bribe, even if they do not have any excess profits, because the higher prices they receive may exceed the additional moral and expense of punishment. Assuming that each firm i delivers a product of a certain quality Y^i and that each firm can vary P^i, then for each firm the admissible set includes such bribes, in which the total "profit" is greater than or equal to zero:

$$0 \le P^i q - T^i - X^i - D^i\left(P^i, Y^i, X^i\right) - N^i\left(X^i\right)$$

Function $X_0^i\left(P^i\right)$ in Fig. 1 is a combination for a bribe, giving zero profit for each firm, and the shaded area and function $X_0^i\left(P^i\right)$ is one of the possible forms of permissible sets bribes. For each seller i, the bribe is maximally possible $X_0^i\left(P^i\right)$, at which profit is zero. In a situation of competition when firms operate independently, determination of the winning bidder can be carried out using the following three-stage procedure: first determine the function $X_0^i\left(P^i\right)$, then determine the combination of price – quality, maximize profits $G^i = X_i - J\left(P^i, Y^i, X^i\right) - R\left(X^i\right)$ bureaucrat $G_{\max}^i$, provided that the profit of the company is zero, and in the end the bureaucrat chooses a firm that maximizes its profit, $G_{\max}$.

3. Next, we will consider various cases of fines for both firms and bureaucrats. In terms of the properties of these functions, we study the influence of institutional conditions on the existence and scale of corruption, as well as the case of a dual monopoly that is not considered.

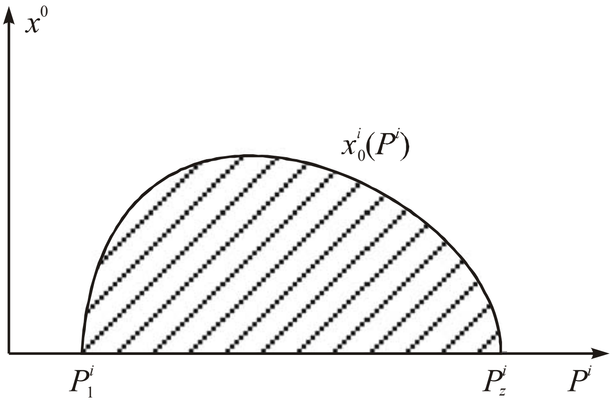

Fig.1 A possible form of admissible bribe value for a firm is possible.

Since the firms that did not win the contract may notify authorities about the bribe, in the interests of the company winning share from competitors loser profit from the contract and may share with them the material costs to support corruption. Ultimately, this leads to a substitute for competitive struggle – the formation of a cartel from former competitors, and the actual competition is replaced by a dual monopoly

Main conclusions. The analysis focuses on:

- the corruption market is a monopoly, oligopolistic or rather competitive company [21]
- costs of members of such an agreement, whether they are physical, moral, which affects their size and how they are related to institutional arrangements – laws, rules, society, traditions, etc.

- what are the social costs of corruption and how to reduce them.

3. Analysis of the limitation of corruption.

Consider the dynamic model of F.T. Louis (F.T. Lui) [6, 7], [22-23] which is a simple model with overlapping generations. It helps to explain why, for example, the level of corruption can seriously increase compared with some periods in the past, while the parameters of the circuit punishment is not too changed. On the other hand, it explains why highly corrupt society conventional measures to combat corruption measures such as enhanced surveillance of bureaucrats is expensive "pleasure" to society, incommensurable compared with the effect on them.

There are two overlapping bureaucrats in the economy in each period: young and old. The number of bureaucrats in two generations is the same. In each period, each bureaucrat is offered a unit of income in the form of a bribe, and he decides whether to accept it or not. If a young bureaucrat accepts a bribe and is subsequently checked, then with the probability of a unit he must pay a fine in C units. He can continue his work in the next period. However, if he takes a bribe again and will be caught, then the new fine will be equal to the C' units already. At the same time C', it is so great that the bureaucrat, who is still punished, will not accept another bribe until the probability of a check is positive. The probability $p(t)$ of checking the bureaucrat during t is the same for everyone.

Bureaucrats in one generation differ only in degrees of their honesty. If the bureaucrat honestly h takes a bribe, then he simply evaluates it into a $1-h$ unit. It is assumed that h- a random variable with a uniform distribution function $F(h),\ h \in [0; 1/f]$. The distribution function $F(h)$ is the same for each generation. It is also assumed that all bureaucrats are at risk.

During a young bureaucrat, he must take into account the expected profit when he becomes old in the period $t+1$. Let probability of verification at the moment $t+1$ expected in t is $p^e(t+1)$. Further, because the punished young bureaucrat actually loses the opportunity to take a bribe in the future, then the young bureaucrat honestly h during t accept a bribe while and only when

$$1-h-p(t)\left[c+\max\left[1-h-p^e(t+1)c,\ 0\right]\right] \geq 0 \quad (5)$$

Because, $\max\left[1-h-p^e(t+1)c,\ 0\right] \geq 0$ then the possible cost of a bribe for the young bureaucrat is no more than the old bureaucrat. This suggests that the latter is more sensitive to corruption than the young one, since the old bureaucrat had not been punished earlier.

Let it $W_0(t+1)=1-p^e(t+1)C$ The young bureaucrat honestly h in t assumes that he will take a bribe $t+1$ when and only when

$$W_0(t+1) \geq h. \quad (6)$$

If (5) is satisfied, then (6) is equivalent

$$1-\frac{p(t)C\left[1-p^e(t+1)\right]}{1-p(t)} \geq h.$$

Let's introduce the designation:

$$\overline{W_y}(t)=1-\frac{p(t)C\left[1-p^e(t+1)\right]}{1-p(t)}.$$

The young bureaucrat honestly h will accept a bribe if and only when

$$\overline{W_y}(t) \geq h.$$

If (6) is not satisfied, then the young bureaucrat in t does not expect to take a bribe in the period $t+1$. Then (5) is equivalent $1-h-p(t)C \geq 0$.

Let it $W_y(t)=1-p(t)C$. The young bureaucrat honestly h will accept a bribe if and only when

$$W_y(t) \geq h.$$

At that, $\overline{W_Y}(t) \leq W_Y(t) \leq W_0(t+1)$ then and only when $p(t) > p^e(t+1)$. It turns out that with $p(t) > p^e(t+1)$ the share of young corrupt bureaucrats in t the assigned function $F\left(\overline{W}_Y(t)\right)$, and with $p(t) \leq p^e(t+1)$ the share of young corrupt bureaucrats t in the given $F\left(\overline{W}_Y(t)\right)$. It is assumed that $p(t) \geq p(t-1)$ if and only if, when $p(t) \geq p(t-1)$, the assumption about the expected change in the probability of the verification turns out to be correct. It is proved that, $p^e(t) \geq p(t-1)$ in proportion to the fear of corrupt bureaucrats t, it is given $(1-p)(t-1)F(W_0(t))$, and in $p^e(t) < p(t-1)$ proportion to the old corrupt bureaucrats in t the $F(W_0)(t)-p(t-1)F\left(\overline{W}_Y(t-1)\right)$ given.

Let $B(t)$ – the part of the corrupted among all the bureaucrats of the generation at the time t. $B(t)$ is an arithmetic mean of the shares of the old and young corrupt bureaucrats taking bribes at the moment t. This value is used to measure the level of corruption in the economy at the time t. The preliminary results can be presented in the following four cases:

$$B(t)=\begin{cases}(1/2)\left[F\left(W_Y(t)\right)+\left(1-p(t-1)\right)\left(F\left(W_0(t)\right)\right)\right],\\ \text{при } p(t)\le p^e(t+1),\ p(t)\le p^e(t+1);\\ (1/2)\left[F\left(\overline{W}_Y(t)\right)+F\left(W_0(t)\right)-p(t-1)F\left(\overline{W}_Y(t-1)\right)\right],\\ \text{при } p(t-1)>p^e(t),\ p(t)>p^e(t+1);\\ (1/2)\left[F\left(W_Y(t)\right)+F\left(W_0(t)\right)-p(t-1)F\left(\overline{W}_Y(t-1)\right)\right],\\ \text{при } p(t-1)>p^e(t),\ p(t)\le p^e(t+1);\\ (1/2)\left[F\left(\overline{W}_Y(t)\right)+\left(1-p(t-1)\right)\left(F\left(W_0(t)\right)\right)\right],\\ \text{при } p(t-1)\le p^e(t),\ p(t)>p^e(t+1).\end{cases} \tag{7}$$

If all proportions are $F(\bullet)$ less than one, then the corresponding expressions for the values W can be subjected to the expression (7). Then we will get

$$B(t)W=(f/2)\left[\left(2-p(t-1)\right)\left(1-p(t)C\right)-J_1+J_2\right] \tag{8}$$

where functions J_1 and J_2 depend on $p(t-1),\ p(t),\ C,\ p^e(t),\ p^e(t+1)$. From (11.8) it follows that it $B(t)$ depends on the probabilities of verification, which are defined below.

At a higher $B(t)$ cost of effective testing is higher. To include this circumstance in the model, the following assumptions are made.

Each time, the government spends R some resources on the check. The resources needed to effectively validate one person at a time t are $r(t)$ available.

It is assumed that

$$r(t)=1/\left(m-nB(t)\right),\ \text{where } m>n>0. \tag{9}$$

Let it be N the total number of bureaucrats. Then

$$p(t)=A-kB(t),\ \text{where } A=Rm/N,\ k=Rn/N. \tag{10}$$

Substituting (10) in (8), we can obtain the regularity of the change $B(t)$. The assumptions made allow us to show that by asking R you can get some stable equilibrium levels of corruption. Let the initial level of corruption in the economy be small. Due to the small cost of checking each person, R it can be spent on more people. Consequently, less people will choose to become corrupt. Similarly, in the opposite case, with a high initial level of corruption.

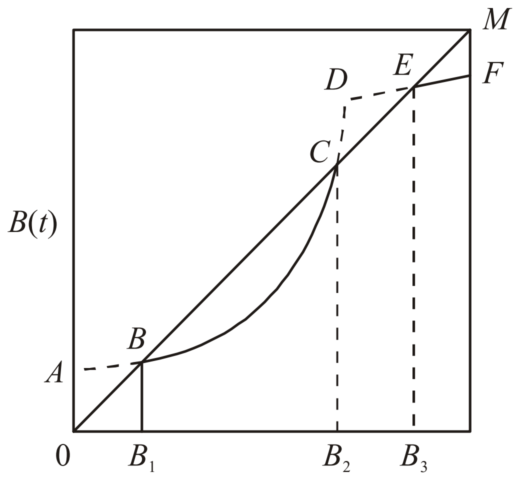

Fig. 2. Phase diagram

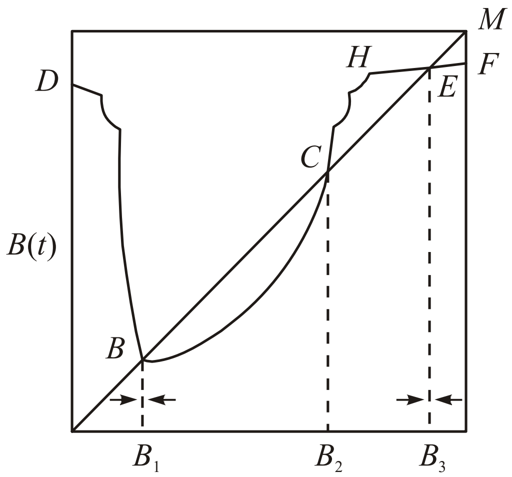

Fig. 3. Phase diagram

It is also assumed that $1 > A > k > 0,\ C > 1 > AC,\ f > 1$. There are three stationary, equilibrium levels of corruption $B^* = B(t)$ for all t. As a result of (9), $p(t) = p^*$ for all t, where p^* is the probability of the corresponding verification B^*. Stationary levels are only possible when $p(t) \le p^e(t+1)$ and $p(t-1) \le p^e(t)$. This case is considered below. Because $F(W_Y(t)) = F(W_0(t)) = f(1-p(t))C$, if $f(1-p(t))C \le 1$ and $F(W_Y(t)) = F(W_0(t)) = 1$, if $f(1-p(t))C > 1$ that

$$B(t) = \begin{cases} (f/2)(2-p(t-1))(1-p(t)c), if \ \_f(1-p(t)C) \le 1, \\ 1-(1/2)p(t-1), \text{ if } \_f(1-p(t)C) > 1. \ (51) \end{cases}$$

The solution can be represented on the phase diagram. In fig. Figure 2 shows a change $B(t)$ chart from $B(t-1)$. The ABCD curve corresponds to (11). It crosses the OM line (this line has a slope of 45%) at two points, B and C. A direct DF representing equation (12) intersects the OM line at point E. Thus, there are three points of equilibrium and it can easily be shown that that only at points B and E it is stable.

If we change the assumption of the correlation between probability of verification and their expected values, then, instead of Fig. 2 we get pic.3. From the figure it is clear that if the initial value of the variable $B(t-1)$ is greater than B_2, or if $B(t-1)$ it is so small that $B(t)$ it is above the point C, it $B(t)$ will climb to the point E. In other cases $B(t)$, it will climb to point B.

Various papers [6] investigate how equilibrium levels of corruption depend on model parameters, and emphasizes the difference between small and significant changes in parameters, because their effects vary.

It is alchebraically more convenient to deal with the stationary probability of verification p^* than with B^*. From equation (10) p^* is related to the stationary equilibrium level of corruption B^* by the following equation:

$$p^* = A - kB^*. \tag{13}$$

From Fig. 3 it follows that there are three possible states of equilibrium for $B(t)$. This is E, C, and D. The first two are determined from the equation

$$Cfkp^*2 - \left[fk(2C+1)\right]p^*(2fk - A) = 0.$$

It is easy to see from (13) that B^* is negatively connected with P^*, therefore, the results obtained from the last equation can be interpreted as follows. If fine C or resources for checking R grows slightly, then B^* it falls. On the other hand, if the average level of honesty h in the economy drops, then f and B^* they rise, which seems natural.

Because apart from point D, point E is also a stationary solution, it is interesting to investigate its dependence on the parameters. It follows from (12) that at the point E

$$B^* = (2-A)/(2-k).$$

This level B^* does not depend on the size of fine C. However, when the resources on the test R increase, then A, and k grow in equal proportions, and B^* therefore falls. So, for an economy with a high level of corruption, changes in R can reduce the level of corruption, and small changes in C can not.

Of particular interest is the question of the transition of the economy from one level of corruption to another. The study of such a question suggests that when society becomes more lenient to corrupt bureaucrats, a sharp increase in corruption can be possible. Moreover, once appearing, the high level of corruption remains, even if the parameters of the restraint scheme will return to the previous level. This explains the existence of societies with levels of corruption, which are sharply different, under the same restrictive schemes. An intuitive explanation for this fact is that, once encountered, corruption requires higher costs for verification and deterrence. Government efforts are becoming less effective.

In addition, it follows from the analysis that, because of the possibility of transition from one equilibrium state to another, sometimes a heavy limiting scheme that seemed inappropriate in a short period, becomes optimal in the long run. At the same time, in some cases, the rough scheme (for example, the introduction of high fines C) can cause a reverse effect, exceeding the economy at a low level of corruption (at point B), to a high level (to the point E). This will happen if corruption "slips" at some point an unstable stationary state (point C) due to fluctuations that arose during the transition.

Concluding Remarks

To make a strict and narrow definition of corruption that restrict corruption to particular agents, sectors or transactions, like corruption as a deviation from the formal rules that regulate the behaviour of public officials, can be handy for fighting corruption when the problem is limited. But because narrow (legal) definitions may ignore vital parts of the problem, like the lack of political will to curb corruption in certain regimes, broader and more open-ended definitions, like corruption in terms of power abuse, will have to be applied to address the situations of pervasive and massive corruption. Even when the effect of democratisation in curtailing corruption is still much debated and not very strong according to available statistics, one basic and practical argument is that corruption can only be reversed by democratising the state. Economic and political competition, transparency and accountability, coupled with the democratic principles of checks and balances, are the necessary instruments to restrict corruption and power abuse.

Conclusions

In the conditions of corruption of all systems of state power on a large scale, a shadow economy can exist and develop, which in turn generates corruption. The economy of Ukraine is characterized by the presence of a rather large amount of the shadow sector, which "undermines" the effectiveness of state mechanisms of stimulating the economy, distorting the conditions for conducting economic activity. At the same time, it should be noted the ambiguity of the influence of the shadow sector on the functioning of the Ukrainian economy in the present. However, the fact remains that the significant amount of the shadow economy is a factor in deepening imbalances existing in the economy, remaining one of the greatest challenges to the country's economic security, the trends of which in future will determine scenarios for the development of the country's economy as a whole. The correctness of this thesis is confirmed by a stable reciprocal relationship between indicators of levels of economic security and the shadow economy. According to various estimates, only direct losses of the state budget of Ukraine from corrupt actions of officials amount to more than 1 billion UAH. The official Gross Domestic Product of Ukraine for the 1st quarter of 2017 amounted to UAH 821277 million. According to experts from the Ministry of Economic Development and Trade of Ukraine [9], the volume of the shadow economy in the 1st quarter of 2017 reaches 37% of the official GDP, while the "shadow" GDP is approximately 30,383 billion UAH, while the total GDP is 1125150 million UAH.

Thus, corruption becomes a systemic threat to economic security, since its distribution is a major obstacle to raising the standard of living of the population, innovative development of the economy, and the formation of a stable society. The growth of corruption, its devastating effect on the country's economy, indicates that there are no effective countermeasures that are necessary to ensure Ukraine's economic security. The consequences of low results in the fight against corruption are the growing threat to the country's economic security. According to the Director of the World Bank, one of the three key areas of reform in our country, through which the Ukrainian economy will receive a growth potential of 4% per year, is to overcome corruption.

Corruption in one form or another is an integral part of human life, corruption is changing, and corruption remains and adapts to new forms of existence. In our time of total informatization (cybernetization) of the world there are new types of corruption – so-called cybercrime, which is a qualitatively new negative aspect of social life. The anti-corruption strategy developed by Transparensy International states that prevention of corruption is more effective and more desirable than "treating" it with force methods. Measures to prevent corruption, as enshrined in the UN Convention against Corruption [8] and presented in various anti-corruption programs, give priority to measures aimed at ensuring transparency and access to information. Therefore, the fight against the negative impact of corruption on the economy can be based on the development and improvement of an open information society (open internet space).

Consequently, the economic and mathematical models of corruption, based on the comprehensive study of the essence of corruption, the causes of its occurrence and sustainable reproduction, taking into account the circumstances that determine the peculiarities of the development of a country, can reveal the important properties of this phenomenon, and require further development to justify the means and methods that help to overcome corruption and increase the level of economic security of Ukraine.

References:

1. Stratehiya natsional'noyi bezpeky Ukrayiny. Zatverdzhena Nakazom Prezydenta Ukrayiny. [Elekronnyy resurs]: – Rezhym dostupu: http://zakon2.rada.gov.ua/laws/show/287/2015.(National Security Strategy of Ukraine. Approved by Order of the President of Ukraine. [Electronic resource]: – Access mode: http://zakon2.rada.gov.ua/laws/show/287/2015.)
2. Năstase C. Economic cooperation – a vector of security and welfare. In: Perspectives of security and defense in Europe. Annual Anniversary Session of Scientific Communications with International Participation 19-20 November 2009, Vol. 1. Bucharest: National Defense University "CAROL I" Publishing House, – 2009. – P. 87-90.
3. Reytynh koruptsiyi E&Y: Ukrayina na pershomu mistsi sered 41 krayiny. [Elektronnyy resurs]: – Rezhym dostupu: http://news.liga.net/ua/news/politics/14730211(Corruption rate E & Y: Ukraine ranked first among 41 countries. [Electronic resource]: – Access mode: http://news.liga.net/ua/news/politics/14730211)
4. Natsional'ne doslidzhennya "Transparency International – Ukrayina". [Elektronnyy resurs]: – Rezhym dostupu: http://www.pravda.com.ua/news/2017/01/25/7133397/(National study "Transparency International – Ukraine". [Electronic resource]: – Access mode: http://www.pravda.com.ua/news/2017/01/25/7133397/)
5. Rose-Ackerman S. The Economics of Corruption // Journal of Political Economy. – 1975. – N 4.
6. Lui F. T. A Dynamic Model of Corruption Deterrence // Journal of Political Economy. – 1996. – N 31.
7. Lui F. T. An Equilibrium Queuing Model of Bribery // Journal of Political Economy. – 1985. – Vol. 93. – N 4.
8. Конвенція ООН проти корупції [Електронний ресурс]. – Режим доступу: http://zakon5.rada.gov.ua/laws/show/995_c16.
9. Загальні тенденції тіньової економіки в Україні у I кварталі 2017 року [Електронний ресурс]. – Режим доступу: http://www.me.gov.ua/Documents/List?lang=uk-UA&id=e384c5a7-6533-4ab6-b56f-50e5243eb15a&tag=TendentsiiTinovoiEkonomiki
10. Alam M. S. (1990) Some Economic Costs of Corruption in LDCs. The Journal ofDevelopment Studies, 27(1), 85-97.
11. Allen D. W. (1991) What Are Transaction Costs? Research in Law and Economics, 14(3), 1-18.
12. Banfield E. C. and Wilson J. Q. (1967) City Politics. Cambridge: Cambridge University Press.
13. Banfield E. C. (1975)/ Corruption as a feature of governmental organisation. The Journal of Law and Economics, 18(3), 587-605.
14. Barzel Y. (1989)/ Economic Analysis of Property Rights. Cambridge: Cambridge University Press.
15. Barzel Y. (2002)/ A Theory of the State. Economic Rights, Legal Rights, and the Scope of the State. Cambridge: Cambridge University Press.
16. Benson B. L. (1990)/ The Enterprise of Law. Justice Without the State. San Francisco: Pacific Research Institute for Public Policy.
17. Benson B. L. and Baden J. (1985)/ The Political Economy of Governmental Corruption: the Logic of Underground Government. Journal of Legal Studies, 14(2), 391 – 410.
18. Coleman J. S. (1990). Foundations of Social Theory, Cambridge (Mass)-London:The Belknap Press of Harvard University Press.
19. Coleman J. W. (1987)/ Toward an Integrated Theory of white Collar Crime. American Journal of Sociology, 93(2), 406-439.
20. Cornish D. (1994)/ Crime as Scripts, in Proceedings of the International Seminar on Environmental Criminology and Crime Analysis, edited by D. Zahm, and P. Cromwell.
21. Coral Gables, University of Miami, 30-45.
22. Cox G., McCubbin M. (2001). Structure and policy: The institutional determinants of policy outcomes, in S. Haggard,M. Mc Cubbins (eds.) The structure of fiscal and regulatory policy,Wahington,World bank, Policy research.
23. Crouch C. (2011). The Strange Non-death of Neo-liberalism, Malden, Polity.

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И. А. Джалладова, д-р физ.-мат. наук, проф.,
В. К. Галицин, д-р экон. наук, проф.,
А. И. Бабинюк, канд. екон. наук
Киевский национальный экономический университет имени Вадима Гетьмана, Киев, Украина

СИСТЕМНЫЙ АНАЛИЗ КОРРУПЦИИ КАК УГРОЗЫ ЭКОНОМИЧЕСКОЙ БЕЗОПАСНОСТИ В КИБЕРПРОСТРАНСТВЕ ЕС

Статья посвящена вопросам моделирования коррупции как угрозы экономической безопасности государства, исследованию зависимости равновесных уровней коррупции от параметров модели. Акцентируется внимание на отличии последствий при небольших и значительных изменениях параметров.

Ключевые слова: коррупция, экономическая безопасность, динамическая модель коррупции, равновесные состояния.

І. А. Джалладова, д-р фіз.-мат. наук, проф.,
В. К. Галіцін, д-р екон. наук, проф.,
О. І. Бабинюк, канд. екон. наук
Київський національний економічний університет імені Вадима Гетьмана, Київ, Україна

СИСТЕМНИЙ АНАЛІЗ КОРУПЦІЇ ЯК ЗАГРОЗИ ЕКОНОМІЧНОЇ БЕЗПЕКИ В КІБЕРПРОСТОРІ ЕС

Стаття присвячена питанню моделювання корупції як загрози економічної безпеки держави, дослідженню залежності рівноважних рівнів корупції від параметрів моделі. Акцентується увага на відмінності наслідків при малих і значних змінах у параметрах.

Ключові слова: корупція, економічна безпека, динамічна модель корупції, рівноважні стани.

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B. Kiforenko, Full Doctor, prof., Ya. Tkachenko, PhD
S. P. Timoshenko Institute of Mechanics of NASU, Kyiv, Ukraine,
I. Vasil'ev, PhD
Taras Shevchenko National university of Kyiv, Kyiv, Ukraine
e-mail: bkifor@ukr.net, e-mail: yaroslavvt@ukr.net, e-mail: igor_v@univ.kiev.ua

OPTIMIZATION OF TRANSFERS BETWEEN CLOSE ELLIPTIC ORBITS WITH SHADOW ACCOUNT

The problem of optimal transfer of an orbital transfer vehicle (OTV) with a solar electric propulsion system between close elliptical orbits during one revolution around the gravitational center was solved. The goal of optimization is the maximum mass of the payload of the OTV at a given initial mass, which is a permanently urgent task of modern space exploration. The influence of the duration and location of the shadow arc of the trajectory on the deterioration of the quality criterion of the problem was estimated. It is revealed that an increase in the duration of a shadow arc leads to a reduction in the weight of the payload. The most negative influence of the shadow is observed in the case when the arc of the trajectory with a relatively large value of the vector of the desired jet acceleration (optimal reactive acceleration, realized in the absence of a shadow) enters the shadow zone.

Keywords: apparatus, orbit, acceleration, payload.

INTRODUCTION. The classic electric propulsive system for spacevehicles consists of two main modules: a power source for powering the engine and an engine itself that generates reactive thrust [1]. To date, two main sources of power of the main electric propulsion system (EP) of interorbital vehicles are considered: nuclear reactor with heat energy conversion into electric and solar cells (panels). In spite of both the ability of a nuclear power source to stay in the Earth's shadow, and higher mass-energy characteristics, in view of the environmental safety requirements, solar cells remain the main variant of the energy supply of the conventional EP when performing inter-orbital transport operations in the near-Earth space [2]. EP with the solar cell energy source will be referred to as the Solar Electric Propulsion system (SEP).

The problem of maximizing the mass of the payload of the Orbit Transfer Vehicle (OTV), which performs orbital maneuvering, is one of the main problems of modern spaceflight mechanics. This necessitates the development of methods for evaluation of transfer trajectories, the motion along which provides a maximum payload mass, with simultaneous optimization of both operating modes and SEP parameters. Transfer trajectories optimization method is proposed in [6] for the case of distant elliptical orbits for an OTV with nuclear EP (NEP), or for an OTV with SEP, if the transfer trajectory does not contain shadow parts. This method uses the solution of the problem of optimal transfer between close elliptical orbits during one revolution around the gravity center [4]. An important property of Electric Rocket Engines (ERE) is the ability to change the operating modes and the corresponding engine parameters within a sufficiently wide range. This allows for theoretical studies to use the mathematical model of so called “ideal” rocket engine of limited power. Optimal control problems for spacecrafts equipped with ideal low thrust rocket engines of limited power are considered in the works [1,5,7,8].

The OTV's interorbit transfer is considered as the motion of material point of the variable mass under the action of the central gravity field and the thrust of an ideal ERE of limited power, which is feeded by electric energy from solar panels. That's why the ERE turns off (engine thrust is zero) in the shadow arcs of the transfer trajectory, and an uncontrolled motion of the spacecraft in the gravitational field takes place. The positions of both the entry and exit points in the shadow are assumed to be given. The mathematical model of the OTV motion as a material point of a variable mass point is adequate enough for the optimal control problems, arising while considering near Earth interorbit transfers.

The purpose of this work is to obtain optimal programs of the vector of reactive acceleration for the transfers of OTV, equipped with ideal SEP, between close elliptical orbits and analysis of shadow impact on the efficiency of SEP's work.

PROBLEM FORMULATION. The initial mass of the OTV M_0 consists of the mass of the payload M_π, the mass of the engine M_γ, the mass of the energy source M_ν and the mass of the rocket propellant M_F required to perform the specified dynamic maneuver:

$$M_0 = M_\pi + M_\gamma + M_\nu + M_F$$

The masses of the engine and energy sources are considered to be proportional to the maximum power of the energy source N_0 [1]:

$$M_\nu = \alpha N_0,\ M_\gamma = \gamma N_0,$$

where the constants α and γ -specific masses of the energy source and the engine are given by assumption.

The problem of maximizing the OTV's payload mass, performing an arbitrary maneuver, when using a model of an ideal ERE may be divided into the trajectory and parametric subproblems [1]. The analytical solution of the parametric problem has the form

$$\begin{gathered} M_\pi = M_0\left(\sqrt{\Phi}-1\right)^2,\ M_\gamma = \frac{M_0\varepsilon\left(\sqrt{\Phi}-\Phi\right)}{1+\varepsilon}, \\ M_\nu = \frac{M_0\left(\sqrt{\Phi}-\Phi\right)}{1+\varepsilon},\ M_F = M_0\sqrt{\Phi},\ \Phi = \frac{\alpha r^{*2}}{2T^*}J \end{gathered} \tag{1}$$

where $\varepsilon = \gamma/\alpha$, r^* – the characteristic linear size, T^* – characteristic time, which is equal to the period of rotation in a circular orbit with a radius of r^* divided into 2π, and J - a functional, which is calculated by the formula

$$J = (1+\varepsilon)\int_0^T\left(W_1^2+W_2^2+W_3^2\right)dt\ . \tag{2}$$

Here T denotes duration of the maneuver, $W_k\ (k=\overline{1,3})$ – components of the jet acceleration vector. Formula (2) and all the subsequent relations are dimensionless: linear dimensions are factorized by r^*, time t is factorized by T^*, and the acceleration – by the value of acceleration of free fall at a distance r^* from the gravity center. From formulas (1) and (2) it follows that the maximum value of the payload mass M_π corresponds to the minimum value of the functional (2).

The trajectory part of the general problem is to determine the OTV's center of mass transfer trajectory, the movement along which provides the minimum value of the functional (2). The equations of the OTV's motion will be written in the osculating variables [3]

$$\begin{gathered} \frac{dx_j}{dt} = \exp\vartheta\sum_{k=1}^{3}F_{jk}W_k, \\ \frac{dE}{dt} = F_{60} + \exp\vartheta\sum_{k=1}^{3}F_{jk}W_k,\ j=\overline{1,5},\quad k=\overline{1,3}, \end{gathered} \tag{3}$$

where $\vec{x} = [\vartheta, e, \omega, I, \Omega]^T$ is a vector whose components are the following set of orbital parameters: the natural logarithm of the angular momentum $\vartheta = \ln\sqrt{a(1-e^2)}$ (here a is semi-major axis), e – eccentricity, ω – pericenter angular distance to the ascending node, I – the orbit plane inclination, Ω – longitude of the ascending node, E – eccentic anomaly, and the coefficients F_{jk} are determined by the formulas

$$\begin{aligned} & F_{11}=0; F_{12}=\frac{1-e\cos E}{1-e^2}; F_{13}=0; F_{21}=\sqrt{1-e^2}\,\frac{\sin E}{1-e\cos E}; F_{22}=\cos E+\frac{\cos E-e}{1-e\cos E}; \\ & F_{23}=0; F_{31}=-\frac{\cos E-e}{e\left(1-e\cos E\right)}; F_{32}=\frac{2-e^2-e\cos E}{e\sqrt{1-e^2}\left(1-e\cos E\right)}\sin E; \\ & F_{33}=-ctgI\left(\frac{\sin\omega}{1-e^2}\left(\cos E-e\right)+\frac{\cos\omega}{\sqrt{1-e^2}}\sin E\right); \\ & F_{41}=F_{42}=0; F_{43}=\frac{\cos\omega}{1-e^2}\left(\cos E-e\right)-\frac{\sin\omega}{\sqrt{1-e^2}}\sin E; F_{51}=F_{52}=0; F_{53}=-\frac{1}{\cos I}F_{33}; \\ & F_{60}=\frac{\left(1-e^2\right)^{\frac{3}{2}}}{\exp(3\vartheta)\left(1-e\cos E\right)}; F_{61}=\left(\frac{\cos E-e}{e\left(1-e\cos E\right)}-1\right)\frac{1}{\sqrt{1-e^2}}; F_{62}=\frac{2-e\cos E}{e\left(1-e\cos E\right)}; \\ & F_{63}=0. \end{aligned} \tag{4}$$

In this paper we consider transfers between close elliptical orbits during one revolution around the gravitational center, therefore the initial and boundary conditions for the system (3) will look like:

$$\begin{aligned} &\vartheta(0)=\vartheta_0, \quad \vartheta(T)=\vartheta_0+\Delta_\vartheta,\\ &e(0)=e_0, \quad e(T)=e_0+\Delta_e,\\ &\omega(0)=\omega_0, \quad \omega(T)=\omega_0+\Delta_\omega,\\ &I(0)=I_0, \quad I(T)=I_0+\Delta_I,\\ &\Omega(0)=\Omega_0, \quad \Omega(T)=\Omega_0+\Delta_\Omega,\\ &E(0)=E_0, \quad E(T)=E_0+2\pi. \end{aligned} \qquad 5)$$

Here $\vartheta_0, e_0, \omega_0, I_0, \Omega_0$ are the components of the initial orbit parameters vector $\vec{x}_0$, E_0 is a value of eccentic anomaly for the point of the initial orbit, where OTV is at the initial time, $\Delta_\vartheta, \Delta_e, \Delta_\omega, \Delta_I, \Delta_\Omega$ – components of the vector of growth of corresponding orbital parameters $\vec{\Delta}$. Since the tranfers between close orbits are considered, therefore the components of the vector $\vec{\Delta}$ are small quantities.

We introduce a function δ_{sh} equal to 1 on the illuminated arcs of the trajectory and 0 on the shaded arcs. To take into account the influence of the shadow on the OTV's motion, the integral function of the functional (2) and all components of the reactive acceleration vector in the equations of motion (3) must be multiplied by δ_{sh}.

As in [4-6], given the smallness of components of the jet acceleration vector (SEP is a low thrust propulsion system), the smallness of the quantities $\Delta_\vartheta, \Delta_e, \Delta_\omega, \Delta_I, \Delta_\Omega$, we linearize the equation of motion (3) and transfer from the independent variable t to the independent variable E. Taking into account the functional (2) and boundary conditions (5), we obtain the next optimal control problem:

$$\begin{aligned} &J=(1+\varepsilon)\int_{E_0}^{E_0+2\pi}\hat{F}_E(E,\vec{x}_0)\left(W_1^2+W_2^2+W_3^2\right)\delta_{sh}dE\to\min,\\ &\frac{dx_j}{dE}=F_E(E,\vec{x}_0)\sum_{k=1}^{3}F_{jk}(E,\vec{x}_0)W_k\delta_{sh}, \quad \vec{x}(E_0)=\vec{x}_0, \quad \vec{x}(E_0+2\pi)=\vec{x}_0+\vec{\Delta}, \end{aligned} \qquad (6)$$

where $\hat{F}_E(E,\vec{x}_0)=1/F_{60}(E,\vec{x}_0), F_E(E,\vec{x}_0)=\exp(\vartheta_0)\hat{F}_E(E,\vec{x}_0)$.

OPTIMAL CONTROL. Using the Pontryagin's maximum principle, we find expressions for optimal jet accelerations and J:

$$\begin{aligned} &W_1=\exp(\vartheta_0)\frac{\lambda_2F_{21}(E,\vec{x}_0)+\lambda_3F_{31}(E,\vec{x}_0)}{2(1+\varepsilon)},\\ &W_2=\exp(\vartheta_0)\frac{\lambda_1F_{12}(E,\vec{x}_0)+\lambda_2F_{22}(E,\vec{x}_0)+\lambda_3F_{32}(E,\vec{x}_0)}{2(1+\varepsilon)},\\ &W_3=\exp(\vartheta_0)\frac{\lambda_3F_{33}(E,\vec{x}_0)+\lambda_4F_{43}(E,\vec{x}_0)+\lambda_5F_{53}(E,\vec{x}_0)}{2(1+\varepsilon)}. \end{aligned} \qquad (7)$$

Here $\vec{\lambda}=[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5]$ is a vector of adjoint functions λ_i of the problem under consieration. Since the coefficients $F_{jk}(E,\vec{x}_0)$ included in the motion equations (the second row in formulas (6)) do not depend on the phase vector $\vec{x}$, then $\lambda_i(E)$ are constant values. The system of motion equations after the substitution of values W_k $(k=\overline{1,3})$ (7) allows integration into elementary functions. The values of the constants λ_i that provide the fulfillment of the initial and boundary conditions of the problem (6) are determined from the system of linear algebraic equations that has a solution

$$\vec{\lambda}=\left(\int_{E_0}^{E_0+2\pi}\mathbf{A}(E,\vec{x}_0)\delta_{sh}dE\right)^{-1}\vec{\Delta}, \qquad (8)$$

where the matrix $\mathbf{A}(E,\vec{x}_0)$ is determined from the relations

$$\mathbf{A}(E,\vec{x}_0)=\exp(\vartheta_0)F_E(E,\vec{x}_0)\hat{\mathbf{A}}\hat{\mathbf{A}}^T/(1+\varepsilon),$$

$$\hat{\mathbf{A}}=\begin{pmatrix} 0 & F_{12}(E,\vec{x}_0) & 0 & 0 & 0\\ F_{21}(E,\vec{x}_0) & F_{22}(E,\vec{x}_0) & 0 & 0 & 0\\ F_{31}(E,\vec{x}_0) & F_{32}(E,\vec{x}_0) & F_{33}(E,\vec{x}_0) & 0 & 0\\ 0 & 0 & F_{43}(E,\vec{x}_0) & 0 & 0\\ 0 & 0 & F_{53}(E,\vec{x}_0) & 0 & 0 \end{pmatrix}.$$

In so doing, the functional J is calculated by the formula

$$J = \frac{1}{2}\vec{\lambda}\vec{\Delta}. \tag{9}$$

Consequently, the relations (6)-(9) is the solution of the optimal control problem (6) with boundary conditions (5).

ESTIMATION OF SHADING INFLUENCE ON SOME ELEMENTARY MANEUVERS CARRYING OUT. Under elementary maneuver, we mean a maneuver in which a small change occurs only in one of the orbital parameters during one revolution around the gravitational center. Fig. 1-3 illustrate the effect of shading on optimal jet acceleration programs when performing elementary maneuvers of the change: the logarithm of the angular momentum ϑ – Fig.1, eccentricity e - Fig.2 and the orbit plane inclination I – Fig.3 by a value of 0.01. Parameters of the initial orbit are assumed to be as follows: $a_0 = 1$, $e_0 = 0.3$, $I_0 = 20^O$, $\omega_0 = \Omega_0 = 0$. The curves 1 in Fig. 1-3 correspond to the fully lit transfer trajectory, curves 2 correspond to the shadow location on the transfer according to formulas (10), and for the curves 3, the shadow location is determined by the formulas (11). The angular distance of the shadow arcs equals to $\Delta_{sh} = 75^O$.

$$\delta_{sh} = \begin{cases} 0, & E \in [\pi - \Delta_{sh}/2,\ \pi + \Delta_{sh}/2], \\ 1, & E \in [0,\ \pi - \Delta_{sh}/2] \cup [\pi + \Delta_{sh}/2,\ 2\pi]. \end{cases} \tag{10}$$

$$\delta_{sh} = \begin{cases} 0, & E \in [0,\ \Delta_{sh}/2] \cup [2\pi - \Delta_{sh}/2,\ 2\pi], \\ 1, & E \in [\Delta_{sh}/2,\ 2\pi - \Delta_{sh}]. \end{cases} \tag{11}$$

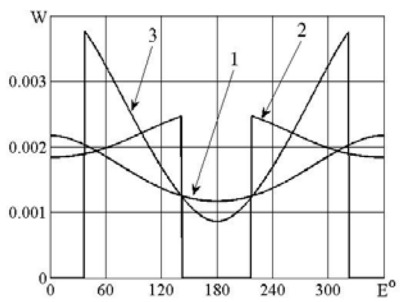

Fig.1. Dependencies W (E) in changing ϑ

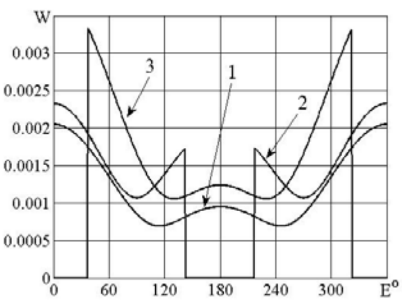

Fig.2. Dependencies W (E) in changing e

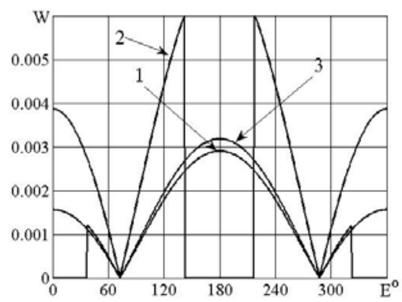

Fig.3. Dependencies W (E) in changing I

An estimate of the propulsion system efficiency is the value of functional J. The smaller the value of J the greater the payload mass (see (1)). Therefore, the negative shadow influence on the effectiveness of a given maneuver is conveniently evaluated with the help of the ratio of the functionals J/J_0 (here J_0 is the functional in the case $\Delta_{sh} = 0$, i.e in the absence of a shadow).

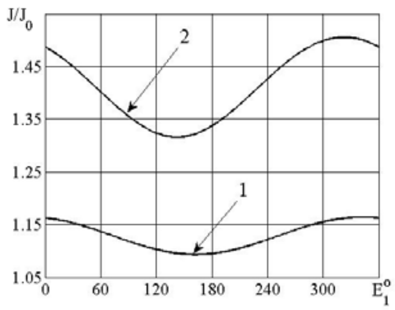

Fig.4. Dependencies J/J₀(E) in changing ϑ

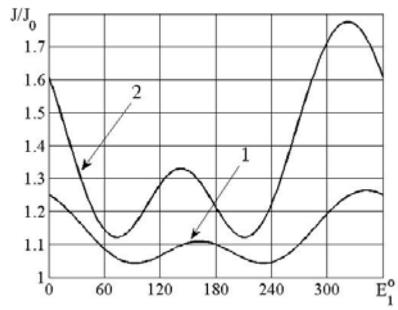

Fig.5. Dependencies J/J₀(E) in changing e

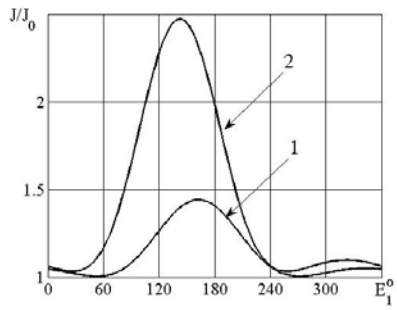

Fig.6. Dependencies J/J₀(E) in changing I

For the maneuvers mentioned above in Figs. 4-6, graphs are given of the dependence of the value J / J_0 from the entry point into the shadow E_1: in case $\Delta_{sh} = 35^O$ – curve 1 and in case $\Delta_{sh} = 75^O$ – curve 2. The function δ_{sh} that determines the distribution of the shadow and illuminated arcs of the trajectory in this case has the following form

$$\delta_{sh} = \begin{cases} \begin{cases} 0, & E \in [E_1, E_1 + \Delta_{sh}], \\ 1, & E \in [0, E_1] \cup [E_1 + \Delta_{sh}, 2\pi], \end{cases} & E_1 + \Delta_{sh} \le 2\pi, \\ \begin{cases} 0, & E \in [0, E_1 + \Delta_{sh} - 2\pi] \cup [E_1, 2\pi], \\ 1, & E \in [E_1 + \Delta_{sh} - 2\pi, E_1], \end{cases} & E_1 + \Delta_{sh} > 2\pi. \end{cases}$$

Analysis of graphs given on the Figs. 1-6 allows to draw the following conclusions. An increase in the angular distance of the shadow arc of the transfer trajectory leads to the OTV's propulsion system efficiency reduction. the location of the shadow arc along the transitional trajectory is essential. Namely, the negative shading influence grows when the arc of the trajectory with a relatively high level of desired reactive acceleration falls into the shadow zone (jet acceleration in the absence of a shadow – curves 1 in Fig. 1-3).

CONCLUSION. The analysis of optimal control of the motion of OTV with the SEP during flight operations between close elliptical orbits showed a negative effect of the shading on the the problem performance index. The results obtained may have, as an independent significance, for example, for the problems of a spacecraft maintaining in the vicinity of a given orbit, and serve as a base for the development of optimal control algorithms for the transfers of OTV with SEP between distant orbits.

References:

1. Grodzovskiy G. L. Spaceflight mechanics. Optimization problems / G. L. Grodzovskiy, Yu. N. Ivanov, V. V. Tokarev. – M. : Nauka, 1975. – 704 p.
2. Electroactive tow for inter-orbit transport of spacecrafts / A. Yu. Zaytseva, V. N. Masley, D. A. Galaburda et al. // Space Science and Technonlogy. – 2015. – 21, № 5. – P. 24–27.
3. Kiforenko B. N. Averaging of Motion Equations in the Problem of Optimization with Respect to Operating Speed of Interorbital Transfer with Thrust of Constant Value in a Strong Central Gravity Field / B. N. Kiforenko, Z. V. Pasechnik, I. Y. Vasiliev // Journal of Automation and Information Sciences. – 2001. – № 6. – P. 62–76.
4. Kiforenko B. N. Some optimal near-elliptical maneuvers of spacecraft with a constant-power propulsion and energy accumulator / B. N. Kiforenko, Ya. V. Tkachenko // Journal of Automation and Information Sciences. –2003 – № 1.– P. 84–100.
5. Kiforenko B. N. Optimization of spacecraft transfers between remote orbits / B. N. Kiforenko, Ya. V. Tkachenko // Applied Mechanics. – 2010. – 46, № 11. – P. 93–100.
6. Kiforenko B. N. The method of optimal transfer trajectories evaluation in a strong central gravitational field / B. N. Kiforenko, Ya. V. Tkachenko // Bulletin of the Institute of Mathematics of NASU. – 2015 – 12, № 3. – P. 60–72.
7. Petukhov V. G. The method of continuation for optimization of interplanetary low thrust trajectories / V. G. Petukhov // Cosmic Research. – 2012. – 50, № 3. – P. 258–270.
8. Petukhov V. G. Optimization of interplanetary trajectories of spacecraft with an ideally regulated engine by continuation method / V. G. Petukhov // Cosmic Research. – 2008. – 46, № 3. – С. 224–237.

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Б. Н. Кіфоренко, д-р фіз.-мат. наук, проф., Я. В. Ткаченко, канд. фіз.-мат. наук
Інститут механіки ім. С. П. Тимошенка НАН України, Київ, Україна,
І. Ю. Васильєв, канд. фіз.-мат. наук
Київський національний університет імені Тараса Шевченка, Київ, Україна

ОПТИМІЗАЦІЯ ПЕРЕХОДІВ МІЖ БЛИЗЬКИМИ ЕЛІПТИЧНИМИ ОРБІТАМИ З УРАХУВАННЯМ ТІНІ

Розв'язано задачу про оптимальний переліт міжорбітального транспортного апарата (МТА) із сонячною електрореактивною рушійною системою між близькими еліптичними орбітами протягом одного оберту навколо гравітаційного центра. Метою оптимізації є максимум маси корисного вантажу МТА при заданій його початковій масі, що є перманентно актуальною задачею сучасної космонавтики. Виконано оцінку впливу тривалості та розташування тіньової дуги траєкторії на погіршення критерію якості задачі. Виявлено, що збільшення тривалості тіньової дуги зумовлює зменшення маси корисного вантажу. Найнегативніший вплив тіні спостерігається у випадку, коли в зону тіні потрапляють дуги траєкторії з відносно великим значенням модуля вектора бажаного реактивного прискорення (оптимального реактивного прискорення, яке реалізується за відсутності тіні).

Ключові слова: апарат, орбіта, прискорення, корисний вантаж.

Б. Н. Кифоренко, д-р физ.-мат. наук, проф., Я. В. Ткаченко, канд. физ.-мат. наук
Институт механики им. С. П. Тимошенко НАН Украины, Киев, Украина,
И. Ю. Васильев, канд. физ.-мат. наук
Киевский национальный университет имени Тараса Шевченко, Киев, Украина

ОПТИМИЗАЦИЯ ПЕРЕХОДОВ МЕЖДУ БЛИЗКИМИ ЭЛЛИПТИЧЕСКИМИ ОРБИТАМИ С УЧЕТОМ ТЕНИ

Решена задача об оптимальном перелете межорбитального транспортного аппарата (МТА) с солнечной электрореактивной двигательной системой между близкими эллиптическими орбитами в течение одного оборота вокруг гравитационного центра. Целью оптимизации является максимум массы полезного груза МТА при заданной его начальной массе, что представляет собой перманентно актуальную задачу современной космонавтики. Проведена оценка влияния продолжительности и расположения теневой дуги траектории на ухудшение критерия качества задачи. Выявлено, что увеличение продолжительности теневой дуги приводит к уменьшению массы полезного груза. Наиболее негативное влияние тени наблюдается в случае, когда в зону тени попадают дуги траектории с относительно большим значением модуля вектора желаемого реактивного ускорения (оптимального реактивного ускорения, которое реализуется при отсутствии тени).

Ключевые слова: аппарат, орбита, ускорение, полезный груз.

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V. R. Kulian, PhD
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine,
O. O. Yunkova, PhD
Vadym Getman National Economics University of Kyiv, Kyiv, Ukraine,

THE PROBLEM OF PORTFOLIO OPTIMIZATION WITH RESTRICTIONS

The problem of constructing analytical methods and computational procedures for solving the two-criterion problem of portfolio optimization is considered. The problem is formulated in the production of G. Markovits, in the presence of quantitative and qualitative instrumental market constraints on the structure of the portfolio.

Keywords: stock portfolio, admissible set, effective set.

INTRODUCTION

For solving and analyzing applied portfolio investment problems there are a wide range of approaches [1], [3]. A significant part of them involves the active use of methods of technical analysis, which make it possible to determine the market value of the stock in the future. Such rules for constructing the forecast, due to the well-developed mathematical formalizations and approaches and relatively not complicated practical implementation, are actively developing and effectively applied not only in the stock market. The use of analytical methods of fundamental analysis allows us to answer the question: why the market value of a stock in the future will be just such? At present, due to the complexity of mathematical models in the study of the market pricing processes of stock market assets, the methods of fundamental analysis have not yet found effective development and constructive application. Principles for analyzing processes based on the development and application of mathematical modeling methods [2], [3] are obviously the most promising and devote much attention to the research.

In this paper an attempt is made to construct new fundamental approaches for solving portfolio investment problems, based on the application of methods of mathematical modeling of dynamic systems and the admissible and effective set of portfolio.

The purpose of the work is to develop analytical methods and computational procedures for solving the problem of two-criterion optimization of a portfolio of risky securities. The problems are presented in the formulation of G. Markovitz in the presence of quantitative and qualitative instrumental market constraints on the structure of the portfolio.

MATHEMATICAL FORMULATION OF THE PROBLEM

The mathematical problem of constructing the optimal dynamics of the portfolio of shares in the most general formulation of G. Markovitz has the form [1]

$$\left\{\begin{array}{l} r^{\mathrm{T}}x = \max\limits_{x} \\ x^{\mathrm{T}}Vx \to \min\limits_{x} \\ I^{\mathrm{T}}x = 1 \\ x_i \geq 0, i = \overline{1,n} \end{array}\right\}. \tag{1}$$

Here $T-$ is the sign of transposition.

The content of this two-criterion task is to determine the optimal investment strategy, which involves maximizing the expected profitability and minimizing the risk at the same time. According to G. Markovitz, the criteria in the task are controversial, that is, improving the outcome of one of them leads to deterioration beyond others. In practice, this means that increasing the profitability of a portfolio corresponds to an increase in its riskiness. There are different approaches to solving the problem (1), but they are more academic in nature and difficult to apply to real investment in securities. A step that can bring the problem formulation (1) closer to the practical investment needs to divide this two-criterion problem into two one-criterion ones. The first of them involves risk optimization at a predetermined level of expected profitability at the chosen time point, and the second is the optimization of the expected profitability for the investor-defined " optimal " portfolio risk level. In some cases, such mathematical statements of nonlinear programming problems allow for analytical solutions [2], but do not consider the essential features. They consist in the fact that at each step the solution of the problem of diversification of the portfolio of portfolios must be taken into account as budgetary and instrumental constraints. The constraints make possible to analyze the availability of the required quantity and quality of financial instruments on the market

$$x_i(t) \in X(t)\,,\, i = \overline{1,n}\,. \tag{2}$$

Here $X(t)-$ a limited set of admissible portfolios. The mathematical formulation of the problem of optimizing the risk of an investment portfolio at a time point determined by the level of its expected profitability is such

$$\left\{\begin{array}{l} r^{\mathrm{T}}(T)x(T) = r_P(T) \\ x^{\mathrm{T}}(T)Vx(T) \to \min\limits_{x} \\ I^{\mathrm{T}}x(T) = 1 \\ x_i(t) \geq 0, i = \overline{1,n}, t \in \left[t_0, T\right] \\ x_i(t) \in X(t), i = \overline{1,n}, t \in \left[t_0, T\right] \end{array}\right\}. \tag{3}$$

On an example of investing in a stock, we consider the problem of optimizing portfolio risk for a given level of its expected return, while taking into account the constraints (2).

THE PROBLEM OF OPTIMAL DIVERSIFICATION OF THE PORTFOLIO

The general task of G. Markovitz of risky assets portfolio optimization (1) involves consideration of another criterion – riskiness. Let's use this in the future by the set of admissible and effective portfolios. At the same time, one of the main tasks of portfolio investment is the problem of optimal diversification of such portfolio.

Let's turn to the second task in the general statement of G. Markovitz about optimization of risk optimal for the expected profitability of the portfolio of shares. To do this, we use the sets of admissible and effective portfolios that correspond to the selected set of shares [1, 2].

The risk optimization procedure for optimal portfolio profitability is to select at each step the permissible portfolios that lie on the line EF . This line connects the point E , that corresponds to the optimal market value of the portfolio with the point, F belonging to the effective set. This line is parallel to the axis of riskiness of portfolios. The feature of this choice of optimal portfolio is that in this line, according to the definition, each of the portfolios corresponds to the same expected profitability, but the risk decreases in the direction of the axis. This property of the admissible set of investment portfolios allows from one side to take into account the limitations (2) and on the other – to determine the portfolio of "optimal" expected returns with less risk.

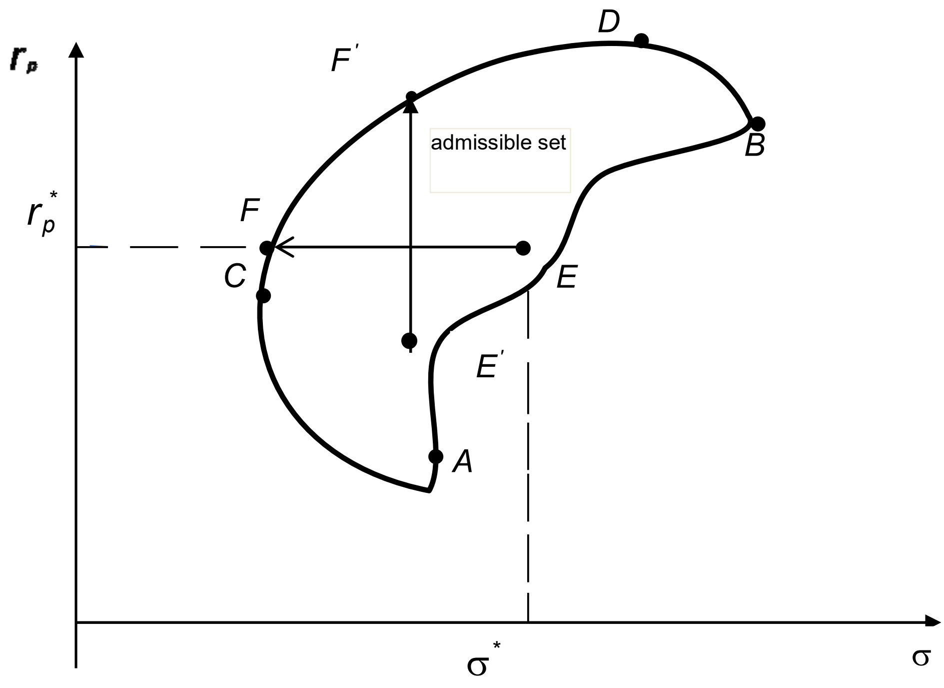

Fig. 1. Solving the problem of optimizing the stock portfolio risk.

If a certain portfolio is at a point E' , that is, for which there is no risk of reducing risk according to the above rule, then we define the "optimal portfolio" by moving it from point E' to point F' that is an element of an effective portfolio of portfolios. In fact, this means the definition of a portfolio of stocks with a higher expected return. At the same time, such a procedure allows constructively to take into account the existing limitations when diversifying the portfolio.

Another mathematical statement of the problem of optimizing the expected return on the investment portfolio at a time point determined by the level of its risk, is such

$$\left\{\begin{array}{l} r^{\mathrm{T}}(T)x(T) \to \max\limits_{x} \\ x^{\mathrm{T}}(T)Vx(T) = \tau \\ I^{\mathrm{T}}x(T) = 1 \\ x_i(t) \geq 0, i = \overline{1,n}, t \in [t_0, T] \\ x_i(t) \in X(t), i = \overline{1,n}, t \in [t_0, T] \end{array}\right\}. \quad (4)$$

The procedure for optimizing the expected return on a portfolio for a certain level of its risk is to select a procedure for optimizing the expected return on a portfolio for a certain level of its risk is to select at each step the admissible portfolios that lie on the straight line EG (Fig. 2). The line connects the points E that corresponds to the optimal expected return on the calculated portfolio and the point G belonging to the effective set . This line is parallel to the axis of market value. The feature of this choice of optimal portfolio is that in this line, according to the definition, each of the portfolios is responsible for the same risk, but the market value increases. This property of the admissible set of investment portfolios, as in the previous case, allows one to take into account the restrictions (2), on the other hand, to define a portfolio with "optimal" risk and higher expected returns.

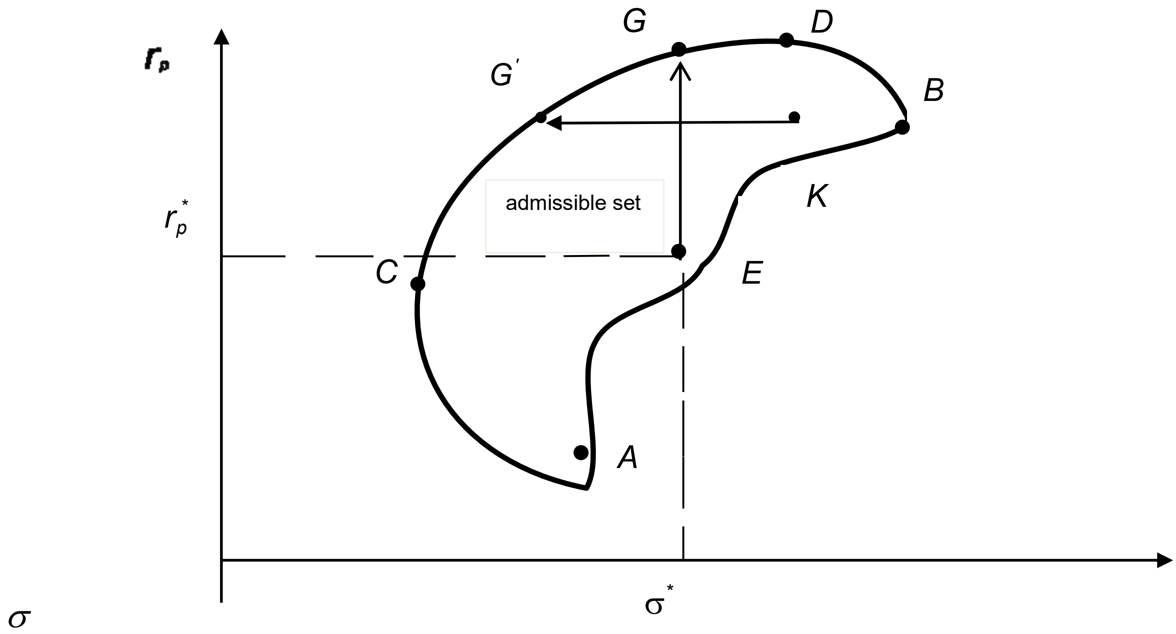

Fig. 2. Solving the problem of optimizing the market value of a portfolio of shares.

As defined above, portfolio is at a point K, that is, for which there is no possibility to increase the expected yield, according to the above rule, then we determine the "optimal portfolio" by moving it from point K to point G' that is an element of an effective portfolio of portfolios. In fact, this means reducing the risk profile of the stock portfolio. Effective set or set of effective portfolios in fig. 1, 2 is on the arc CD. It is a set of Pareto [1] for an existing set of shares on the market.

CONCLUSION

In this study, new mathematical statements of optimization of stock portfolio structure are presented and methods of their solution are developed. Mathematical problems formulated on the basis of models of the dynamics of market value of one share and portfolio of shares. That gives an opportunity to solve the problem of optimal diversification of the portfolio of investments, taking into account quantitative and qualitative market restrictions on the structure of the portfolio.

Reference:

1. *Шарп У.* Инвестиции / *Уильям Ф. Шарп, Гордон Дж. Александер, Джеффри В. Бэйли.* – М. : Инфра-М, 1999. – 1027 с.
2. *Гаращенко Ф. Г.* Качественный анализ математических моделей инвестиционного менеджмента / Ф. Г. Гаращенко, В. Р. Кулян, В. В. Рутицкая // Кибернетика и вычислительная техника. – 2005. – № 148. – С. 3–10.
3. *Garashchenko Fedir G.* Modelling and Analysis of Investment Trends / Fedir G. Garashchenko, Viktor R. Kulian, Vladislava V. Rutitskaya // Journal of Automation and Information. – New-York, Connecticut. – 2011. – V. 43, issue 12. – P. 48–58.

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В. Р. Кулян, канд. техн. наук, доц.
Київський національний університет імені Тараса Шевченка, Київ, Україна,
Е. А. Юнькова, канд. фіз.-мат. наук, доц.
Київський національний економічний університет імені Вадима Гетьмана, Київ, Україна

ЗАДАЧА ОПТИМІЗАЦІЇ ПОРТФЕЛЮ АКЦІЙ ЗА ОБМЕЖЕНЬ

Розглянуто проблему побудови аналітичних методів і обчислювальних процедур для розв'язання двокритеріальної задачі оптимізації портфелю акцій. Задача формулюється в постановці Г. Марковіца за наявності кількісних і якісних інструментальних ринкових обмежень на структуру портфелю.

Ключові слова: портфель акцій, допустима множина, ефективна множина.

В. Р. Кулян, канд. техн. наук, доц.
Киевский национальный университет имени Тараса Шевченко, Киев, Украина,
Е. А. Юнькова, канд. физ.-мат. наук, доц.
Киевский национальный экономический университет имени Вадима Гетьмана, Киев, Украина

ЗАДАЧА ОПТИМИЗАЦИИ ПОРТФЕЛЯ АКЦИЙ ПРИ ОГРАНИЧЕНИЯХ

Рассматривается проблема построения аналитических методов и вычислительных процедур для решения двокритериальной задачи оптимизации портфеля акций. Задача формулируется в постановке Г. Марковица при наличии количественных и качественных инструментальных рыночных ограничений на структуру портфеля.

Ключевые слова: портфель акций, допустимое множество, эффективное множество.

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A. V. Nikitin, PhD Physics and Mathematics, assoc. prof.,
D. P. Proskurin, Dr. Sci. Physics and Mathematics, assoc. prof.,
R. Ya. Yakymiv, PhD Physics and Mathematics, assoc. prof.
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

SUFFICIENT CONDITIONS OF STABILITY IN MEAN SQUARE OF SOLUTIONS TO LINEAR DIFFERENCE EQUATIONS WITH RANDOM COEFFICIENTS

The article is devoted to the study of sufficient conditions for asymptotic stability in the mean square solutions of linear difference equations with Markov coefficients.

Keywords: stochastic difference equation, Markov process, stability in mean square.

Introduction. Tasks of stability of stochastic difference equations were engaged, in particular, K. G. Valjeev [2], J. I. Hikhman, A. V. Skorokhod [1], I. A. Jalladov, D. G. Korenivsky [11], D. Ya. Khusainov [4] and other well-known scholars. various approaches may be used to study the stability of solutions of stochastic equations. So, if known solutions of such equations, then a direct method of studying stability or asymptotic stability is sometimes used. another approach is based on the introduction of lyapunov stochastic functions and the use of the analogue of the second lyapunov method. If known instant functions or equations for moment functions, then the problems of studying stability in the mean square is reduced to the analysis of the behavior of such functions.

In the paper, using the recurrence equations for the moments of the first order of solutions of linear difference equations with random coefficients, sufficient stability conditions in a quadratic one are substantiated.

Consider the Markov process $\xi_k(t)$, which can be in states $\theta_{k1}, \theta_{k2}, ..., \theta_{kn}$ with probabilities $p_{km}(t) = \mathrm{P}\{\xi_k(t) = \theta_{km}\}$ $(k = 1, 2, ..., n)$. Let the sequence of random variables X_n be a solution of a linear stochastic difference equation

$$X_{n+1} = A(\xi_{n+1}, \xi_n) X_n + \sum_{j=1}^{r} B_j(\xi_{n+1}, \xi_n) \eta_{jn} X_n, \quad X_0 = \zeta_0, \tag{1}$$

where the vectors $\eta_k = (\eta_{1k}, ..., \eta_{rk})^*$, $k = 0, ..., n$, are a collection of random variables that do not depend on ξ_k and do not depend on the random vector ζ_0, A, B_j are matrices which have dimension $m \times m$.

Definition 1. The trivial solution of equation (1) is called asymptotically stable in the mean square if

$$\lim_{n \to \infty} \mathbb{E} |\mathrm{X}_n|^2 = 0$$

for an arbitrary deterministic vector X_0.

Partial moments $m_k(n)$ are determined from the equation

$$m_k(\mathrm{n}+1) = \sum_{s=1}^{q} A_{ks} m_s(\mathrm{n}) \pi_{ks},$$

so, for the vector $M(n) = (m_1(n), ..., m_q(n))^*$ can be written the equation

$$M(n+1) = AM(n),$$

where the matrix A can be determined from the relation

$$A = \begin{pmatrix} \pi_{11}A_{11} & \pi_{12}A_{12} & \dots & \pi_{1q}A_{1q} \\ \pi_{21}A_{21} & \pi_{22}A_{22} & \dots & \pi_{2q}A_{2q} \\ \dots & \dots & \dots & \dots \\ \pi_{q1}A_{q1} & \pi_{q2}A_{q2} & \dots & \pi_{qq}A_{qq} \end{pmatrix}.$$

We recall the statement [10], which replaces the problem of stability study with the problem of solvability of the matrix equation.

Statement. In order for the zero solution of equation (1) to be asymptotically stable in the mean square, it is sufficient that there exist positively defined matrices Q_k of dimension $m \times m$ such that there are solutions of equations

$$Q_k - \sum_{s=1}^{q} \pi_{sk} (A_{sk}^* Q_k A_{sk} + \sum_{j=1}^{r} B_{jsk}^* Q_s B_{jsk}) = E, \quad k = \overline{1, q}. \tag{2}$$

2. Iterative scheme for solving the matrix equation.

We propose an iterative scheme for solving the matrix equation (2), which is a modification of the fixed point method. Let us rewrite the equation (2) in the form

$$-\sum_{s=1}^{q}\pi_{sk}\,(A_{sk}^{*}Q_k A_{sk}+\sum_{j=1}^{r}B_{jsk}^{*}Q_s B_{jsk})=E-Q_k\,,\quad k=\overline{1,q}\,. \tag{3}$$

Let's denote

$$F[Q_k]=-\sum_{s=1}^{q}\pi_{sk}\,(A_{sk}^{*}Q_k A_{sk}+\sum_{j=1}^{r}B_{jsk}^{*}Q_s B_{jsk})\,,$$

$$D[Q_k]=E-Q_k$$

and construct an implicit solution of the equation

$$F[Q_k]=D[Q_k]$$

for the next iterative scheme

$$F[Q_{k+1}]=D[Q_k],\ \ k=0,1,2,... \tag{4}$$

We hold sufficient conditions for the convergence of the method (4) to solve the equation (3).

Theorem 2. *Let the following conditions be fulfilled:*

The eigenvalues of the matrix A are negative;

2) $|T^{-1}|\le\beta$ *; де* $T\equiv A^{-1}\otimes E+E\otimes A$ *,* $0<\beta<1$*;*

3) $|E-Q_k|<\alpha Q_k$ *,* $0<\alpha<1$ *.*

Then the sequence $\{Q_k, k\ge 1\}$ *converges with the solution of the equation* (3).

Proof. Since condition 1) is fulfilled, then the operator D that translates the space of positively defined matrices in itself. Matrix equation

$$-\sum_{s=1}^{q}\pi_{sk}\,(A_{sk}^{*}Q_k A_{sk}+\sum_{j=1}^{r}B_{jsk}^{*}Q_s B_{jsk})=K\,, \tag{5}$$

where K – an arbitrary positive definite matrix, also a positively defined matrix is also a solution Q_k , i.e. F^{-1} also translates the space of positively defined matrices in itself (since, by condition 1), all $\lambda_i(\mathrm{A})<0$.

Enter the metric as follows

$$\rho(Q_1,Q_2)=||\,Q_1-Q_2\,||\,.$$

The space of positively defined matrices with the metric thus entered will be a complete metric space. Then for the convergence of the iterative scheme it remains to check the fulfillment of the condition that the operators D і F^{-1} are operators of sequential compression. According to condition 3), we will receive

$$\rho(\mathrm{F}(Q_1),\mathrm{F}(Q_2))<\alpha\rho(Q_1,Q_2)\,.$$

We write the equation (9) in the form

$$TQ_k=K\,,$$

where $T\equiv A^{*}\otimes I+I\otimes A$ – the Kroneker product of the matrices. Then

$$Q_k=T^{-1}K$$

and

$$\rho(F^{-1}(Q_1),F^{-1}(Q_2))=||\,T^{-1}Q_1-T^{-1}Q_2\,||\le\beta\,,$$

which means the fulfillment of the condition of sequential compression.

The theorem is proved.

References:

1. Gikhman I. I. Stochastic differential equations and their applications / I. I. Gikhman, A. V. Skorokhod. – K. : Naukova Dumka, 1982. – 612 p.
2. Valeev, K. G. Stability of Linear Dynamical Systems with Random Parameters / K. G. Valeev. – K. : General "Knowledge" of the Ukrainian SSR, 1986. – 24 p.
3. Dunford N. Linear operators. 1. General theory / N. Dunford, J. T. Schwartz. – Moscow : NL, 1962. – 895 p.
4. Khusainov D. Ya. Estimation of solutions of linear systems with the use of Lyapunov functions / D. Ya. Khusainov, E. V. Ivokhin. – Kiev : Cybernetics, 1985. – № 2. – P. 7–10.
5. Dynkin E. B. Markov processes / E. B. Dynkin. – Moscow : Fizmatgiz, 1963. – 859 p.
6. Katz I. Ya. The method of Lyapunov functions in problems of stability and stabilization of systems of random structure / I. Ya. Katz. – Ekaterinburg : Publishing House of the Ural State Academy of Communications, 1998. – 222 p.
7. Skorokhod A. V. Asymptotic methods of the theory of stochastic differential equations / A. V. Skorokhod. – Kiev : Naukova Dumka, 1987. – 328 p.
8. Khasminsky R. Z. Stability of systems of differential equations for random perturbations of their parameters / R. Z. Khasminsky. – Moscow : Nauka, 1969. – 367 p.
9. Hale J. Theory of functional differential equations / J. Hale. – Moscow : Mir, 1984. – 421 p.
10. Shusharin Yu. V. Algebraic criteria for asymptotic stability of solutions of linear difference equations with random coefficients // Yu. V.Shusharin / Journal of Computational and Applied Mathematics, 2014. – № 3. – P. 167–172.
11. Korenivsky D. G. Stability of solutions of systems of difference equations for stochastic perturbations of their coefficients. Algebraic criteria / D. G. Korenivsky. – Kyiv : Institute of Mathematics of the National Academy of Sciences of Ukraine, 2010. – 210 p.

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А. В. Нікітін, канд. фіз.-мат. наук, доц.,
Д. П. Проскурін, д-р фіз.-мат. наук, доц.,
Р. Я. Якимів, канд. фіз.-мат. наук, доц.
Київський національний університет імені Тараса Шевченка, Київ, Україна

ДОСТАТНІ УМОВИ СТІЙКОСТІ У СЕРЕДНЬОМУ КВАДРАТИЧНОМУ РОЗВ'ЯЗКІВ ЛІНІЙНИХ РІЗНИЦЕВИХ РІВНЯНЬ ІЗ ВИПАДКОВИМИ КОЕФІЦІЄНТАМИ

Досліджено достатні умови асимптотичної стійкості у середньому квадратичному розв'язків лінійних різницевих рівнянь із марковськими коефіцієнтами.

Ключові слова: стохастичне різницеве рівняння, марковський процес, стійкість у середньому квадратичному.

А. В. Никитин, канд. физ.-мат. наук, доц.,
Д. П. Проскурин, д-р физ.-мат. наук, доц.,
Р. Я. Якимов, канд. физ.-мат. наук, доц.
Киевский национальный университет имени Тараса Шевченко, Киев, Украина

ДОСТАТОЧНЫЕ УСЛОВИЯ УСТОЙЧИВОСТИ В СРЕДНЕМ КВАДРАТИЧЕСКОМ РЕШЕНИЙ ЛИНЕЙНЫХ РАЗНОСТНЫХ УРАВНЕНИЙ СО СЛУЧАЙНЫМИ КОЭФФИЦИЕНТАМИ

Исследованы достаточные условия асимптотической устойчивости в среднем квадратичном решений линейных разностных уравнений с марковскими коэффициентами.

Ключевые слова: стохастическое разностное уравнение, марковский процесс, устойчивость в среднем квадратичном.

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D. Ya. Khusainov, Dr. Sci., Professor,
A. V. Shatyrko, Ph.D., Associate Professor
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine,
M. Pokojovy, Ph.D., Assistant Professor
The University of Texas at El Paso, Department of Mathematical Sciences, El Paso, TX, USA

ANALYSIS OF AN EPIDEMIOLOGICAL MODEL WITH AFTEREFFECT

In the present article, a disease dynamics model is investigated. Model factors influencing the process dynamics are analyzed. Stability of the equilibrium states of the system without delay is investigated. Numerical integration of the system for a model example is performed and the phase portrait is plotted. A system with time delay is also considered.

Keywords: differential equations, mathematical model, phase portrait, steady state, systems with delay.

Introduction

First attempts to apply mathematical methods to investigation of disease spread (epidemy) process dynamics probably date back to Bernoulli's investigations perfomed as early as in the middle of the 17th century. He used elementary mathematical instruments to assess the efficiency of pox prevention strategies. The emergence of computers in the 50s of the 20th century marked a new era in mathematical epidemiology giving birth to a new methodology – the so-called epidynamics [3–5]. It was founded upon the scientific analogy in modeling epidemic processes, i.e., the way the infection spreads from infectious individuals to other population members), with diffusion processes (of matter, energy, impulse, etc.) studied in mathematical physics.

The contemporary world is currently facing dangerous circumstances when both old and new inflectional diseases not only have a significant potential to spread, but to spread unprecedently fast [6]. This fact justifies the importance of the problem considered in the present paper.

1. Epidemiological models without delay

1.1. Notations and basic relations

The vast majority of mathematical epidemiogical models are based on the so-called compartment principle which postulates that the population can be disjointly split into a set of classes or categories depending on the way they interact with the disease. The simple model can be formulated as follows [7].

Consider a population of N individuals that can be split into three groups:

$S(t)$ – all individuals that are susceptible to the particular disease, but not infected at time t;

$I(t)$ – all infectious individuals (that are spreading the disease);

$R(t)$ – all individuals that are immune to the disease at time t ('R' usually stands for 'recovered').

This model is referred to as the SIR-model. The interaction among these classes is schematically displayed in Figure 1.1.

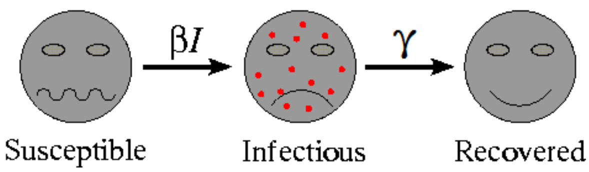

Figure. 1.1. Interactions in the SIR-model [23]

An accurate quantitative description of the model can be obtained using the abstract mathematical language as a system of coupled differential equations for each of the classes. Let $f(S,I)$ stand for the number of individuals newly infected, whose particular form depends on the number of encounters between susceptible $S(t)$ and infectious $I(t)$ individuals. We assume the infection can only be passed from an infectious to a susceptible individual. Then, the system of ordinary differential equations reads as:

$$\frac{d}{dt}S(t) = -f(S(t), I(t)),$$
$$\frac{d}{dt}I(t) = f(S(t), I(t)) - \gamma I(t),$$
$$\frac{d}{dt}R(t) = \gamma I(t), \ \gamma > 0.$$

Here, $\gamma > 0$ denotes the recovery rate. The term $\gamma I(t)$ quantifies the individuals that have formed immunity to the disease.

Typically, $f(S,I)$ has the following properties:

1. $f(S,I) = 0$ for $S = 0$ or $I = 0$,
2. $f(S,I) > 0$ for $S > 0$, $I > 0$,
3. $f_{SS}(S,I) \le 0$, $f_{II}(S,I) \le 0$ for $S > 0$, $I > 0$,
4. $f_S(S,I) > 0$, $f_I(S,I) > 0$ for $S > 0$, $I > 0$.

An example for $f(S,I)$ is $f(S,I) = \beta SI$, where β is a constant parameter describing the chance of getting infected from an encounter with an infected individual. This yields the system

$$\frac{d}{dt}S(t) = -\beta S(t)I(t),$$
$$\frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t), \tag{1.1}$$
$$\frac{d}{dt}R(t) = \gamma I(t), \ \gamma > 0.$$

Summing up the equations, we easily obtain

$$\frac{d}{dt}(S(t) + I(t) + R(t)) = 0.$$

Hence, the first integral of the system is

$$S(t) + I(t) + R(t) = \text{const}.$$

This corresponds to the situation when there is no mortality.

Numerical solutions to SIR-equations are in good correspondence with the theory of oscillatory processes. Indeed, despite of the high epidemiological activity on a compact time horizon, the infection level eventually stabilizes at a particular constant level (Figure 1.2).

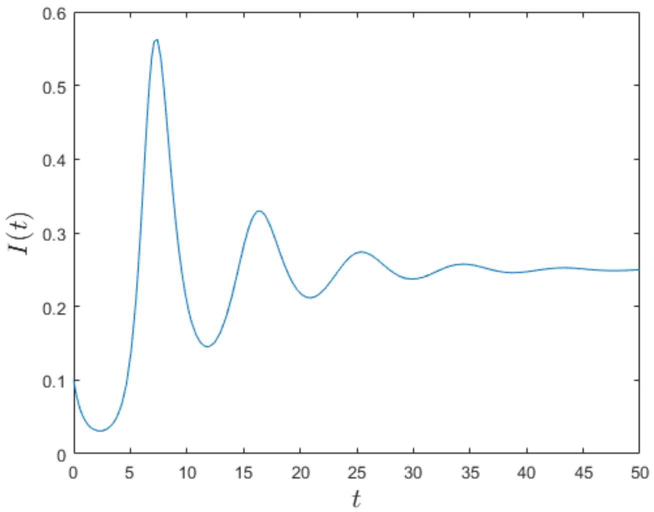

Figure 1.2. Decaying oscillations in the SIR-model

Further investigations of epidynamics processes pursued the path of improving mathematical models to better match the reality of these proceses (particular disease types and their specifics, etc): nonlinear factors, delay effects, stochastic aspects, etc. [8–13].

1.2. A generalized model

We make the model in Equation (1.1) more comprehensive by introducing a constant birth intensity $\lambda > 0$ to the first equation. The terms $-\mu S(t)$, $-\mu I(t)$, $-\mu R(t)$ are further introduced to model the mortality. Here, $\mu > 0$ is the mortality constant [14–16]. This yields:

$$\frac{dS(t)}{dt} = -\beta S(t)I(t) + \lambda - \mu S(t), \quad \frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t), \quad \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t). \tag{1.2}$$

Here, all parameters are positive, i.e., $\lambda > 0$, $\mu > 0$, $\beta > 0$, $\gamma > 0$. Adding up the equations, we obtain the linear first-order inhomogeneous equation

$$\frac{d}{dt}(S(t) + I(t) + R(t)) = \lambda - \mu(S(t) + I(t) + R(t)),$$

which, being integrated, yields the first integral

$$S(t) + I(t) + R(t) = [S(0) + I(0) + R(0)]e^{-\mu t} + \frac{\lambda}{\mu}[1 - e^{-\mu t}]. \tag{1.3}$$

Obviously,

$$S(t) + I(t) + R(t) \to \frac{\lambda}{\mu} \quad \text{as} \quad t \to +\infty,$$

i.e., the total number of susceptible, infectious and recovered individuals eventually stabilizes.

Since the $R(t)$ – variable does not occur in the first two equations, they can be decoupled from the third one. Hence, we can restrict our investigations to the planar system

$$\frac{dS(t)}{dt} = \lambda - \mu S(t) - \beta S(t)I(t), \quad \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t). \tag{1.4}$$

1.3. Steady states on the plane

Determining the steady states for the autonomous system (1.4) reduces to solving the nonlinear algebraic system

$$\lambda - \mu S - \beta SI = 0, \ \beta SI - (\mu + \gamma)I = 0.$$

Adding up the equations, we get

$$\lambda - \mu S - (\mu + \gamma)I = 0.$$

Thus,

$$S = \frac{\lambda - (\mu + \gamma)I}{\mu}. \tag{1.5}$$

Plugging this into the first equation, we obtain

$$(\mu + \gamma)I - \beta\frac{\lambda - (\mu + \gamma)I}{\mu}I = 0,$$

or, equivalently,

$$\mu(\mu + \gamma)I - \beta[\lambda - (\mu + \gamma)I]I = 0.$$

This is a quadratic equation with two solutions:

$$I_1 = 0, \ I_2 = \frac{\beta\lambda - \mu(\mu + \gamma)}{\beta(\mu + \gamma)} = \frac{\lambda}{\mu + \gamma} - \frac{\mu}{\beta}.$$

Correspondingly, Equation (1.5) also possesses two solutions

$$S_1 = \frac{\lambda - (\mu + \gamma)(\mu + \gamma)I_1(t)}{\mu} = \frac{\lambda}{\mu}, \quad S_2 = \frac{\lambda - (\mu + \gamma)I_2(t)}{\mu} = \frac{\mu + \gamma}{\beta}.$$

Therefore, assuming

$$\beta\lambda - \mu(\mu + \gamma) > 0, \tag{1.6}$$

Equation (1.4) has two steady states: $O_1(S_1, I_1)$, $O_2(S_2, I_2)$, both lying in the first quadrant:

$$O_1\left(\frac{\lambda}{\mu}, 0\right), \ O_2\left(\frac{\mu + \gamma}{\beta}, \frac{\lambda}{\mu + \gamma} - \frac{\mu}{\beta}\right).$$

1.4. Steady state investigation

Our investigation will be performed using the linearization method. The linear approximation of the nonlinear system (1.4) in a neighborhood of a steady state $O(S_0, I_0)$ reads as

$$\frac{dS(t)}{dt} = [-\mu - \beta I]\Big|_{(I,S)=(S_0,I_0)}(S(t) - S_0) - \beta S\Big|_{(I,S)=(S_0,I_0)}(I(t) - I_0)$$

$$\frac{dI(t)}{dt} = \beta I\Big|_{(I,S)=(S_0,I_0)}(S(t) - S_0) + [\beta S - (\mu + \gamma)]\Big|_{(I,S)=(S_0,I_0)}(I(t) - I_0). \tag{1.7}$$

Consider each of the steady states $O_1(S_1,I_1)$, $O_2(S_2,I_2)$ separately.

1. Plugging the coordinates of the first steady state $O_1\left(\frac{\lambda}{\mu},0\right)$ into the linear approximation system (1.7), we obtain

$$\frac{dS(t)}{dt}=-\mu\left(S(t)-\frac{\lambda}{\mu}\right)-\lambda\frac{\beta}{\mu}I(t),\ \frac{dI(t)}{dt}=\left[\lambda\frac{\beta}{\mu}-(\mu+\gamma)\right]I(t). \tag{1.8}$$

Hence, the matrix A_1 of the system is given as

$$A_1=\begin{bmatrix}-\mu & -\lambda\frac{\beta}{\mu}\\ 0 & \lambda\frac{\beta}{\mu}-(\mu+\gamma)\end{bmatrix},$$

and its characteristic equation reads as

$$\det\{A_1-\xi E\}=\begin{vmatrix}-\mu-\xi & -\lambda\frac{\beta}{\mu}\\ 0 & \lambda\frac{\beta}{\mu}-(\mu+\gamma)-\xi\end{vmatrix}=[-\mu-\xi]\times[\lambda\frac{\beta}{\mu}-(\mu+\gamma)-\xi]=0.$$

Solving this algebraic equation, we get the eigenvalues

$$\xi_1=-\mu<0,\ \xi_2=\lambda\frac{\beta}{\mu}-(\mu+\gamma).$$

On the strength of condition (1.6), $\xi_2>0$. Thus, the steady state is a saddle point. Substituting the straight line equation $\overline{I}(t)=k\overline{S}(t)$ into the linear system (1.8) with the shifted origin $I=I_1+\overline{I}$, $S=S_1+\overline{S}$, we obtain

$$\frac{d\overline{I}(t)}{d\overline{S}(t)}=\frac{\left[\lambda\frac{\beta}{\mu}-(\mu+\gamma)\right]\overline{I}(t)}{-\mu\overline{S}(t)-\beta\frac{\lambda}{\mu}\overline{I}(t)}.$$

Thus,

$$k=\frac{\left[\lambda\frac{\beta}{\mu}-(\mu+\gamma)\right]k}{-\mu-\lambda\frac{\beta}{\mu}k}.$$

This equation has two distinct roots: $k_1=0$, $k_2=\frac{\mu\gamma}{2\beta\lambda}$. This yields two separatrices: $I(t)=0$, the unstable one, and $I(t)=kS(t)$, the stable one, with $k=\frac{\mu\gamma}{2\beta\lambda}$.

2. Substituting the coordinates of the second steady state $O_2\left(\frac{\mu+\gamma}{\beta},\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)$ into the linear approximation in Equation (1.8), we arrive at

$$\frac{dS(t)}{dt}=-\lambda\frac{\beta}{\mu+\gamma}\left(S(t)-\frac{\mu+\gamma}{\beta}\right)-(\mu+\gamma)\left(I(t-\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)\right),$$

$$\frac{dI(t)}{dt}=\beta\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)\left(S(t)-\frac{\mu+\gamma}{\beta}\right). \tag{1.9}$$

Therefore, the coefficient matrix A_2 reads as

$$A_2=\begin{bmatrix}-\lambda\frac{\beta}{\mu+\gamma} & -(\mu+\gamma)\\ \lambda\frac{\beta}{\mu+\gamma}-\mu & 0\end{bmatrix}$$

with the characteristic equation

$$\det\{A_2-\xi E\}=\begin{vmatrix}-\lambda\dfrac{\beta}{\mu+\gamma}-\xi & -(\mu+\gamma)\\ \lambda\dfrac{\beta}{\mu+\gamma}-\mu & -\xi\end{vmatrix}=\xi^2+\lambda\frac{\beta}{\mu+\gamma}\xi+[\beta\lambda-\mu(\mu+\gamma)]=0.$$

Since all coefficients in this quadratic equation are positive, on the strength of Routh & Hurwitz criterion, the steady state is asymptotically stable. The eigenvalues are

$$\xi_{1,2}=\frac{1}{2}\left\{-\frac{\lambda\beta}{\mu+\gamma}\pm\sqrt{\left(\frac{\lambda\beta}{\mu+\gamma}\right)^2-4[\beta\lambda-\mu(\mu+\gamma)]}\right\}.$$

Assuming $\beta\lambda-\mu(\mu+\gamma)>0$ (which is equivalent with the assumption that the steady state lies in the first quadrant), the steady state $O_2\left(\frac{\mu+\lambda}{\beta},\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)$ is asymptotically stable. Thus, for $\left(\frac{\lambda\beta}{\mu+\gamma}\right)^2-4[\beta\lambda-\mu(\mu+\gamma)]>0$, we obtain a stable node (sink) and for $\left(\frac{\lambda\beta}{\mu+\gamma}\right)^2-4[\beta\lambda-\mu(\mu+\gamma)]<0$ is a stable focus.

The complete phase portrait of the nonlinear system is obtain by 'gluing' each of the separate phase portraits together.

Example 1. Consider the following parameter selection: $\lambda=2$, $\beta=2$, $\mu=1$, $\gamma=1$. Then, the system has the form

$$\frac{dS(t)}{dt}=2-S(t)-2S(t)I(t),\quad \frac{dI(t)}{dt}=2S(t)I(t)-2I(t).$$

The steady states are $O_1(2,0)$, $O_2\left(1,\frac{1}{2}\right)$. Consider the first steady state. The coefficient matrix and its eigenvalues in $O_1(2,0)$ are

$$A_1=\begin{bmatrix}-1 & -4\\ 0 & 2\end{bmatrix},\ \lambda_1(A_1)=-1,\ \lambda_2(A_1)=2.$$

The equilibrium is a saddle with two separatrices: $k_1=0$ (the stable one) and $k_2=-\frac{3}{4}$ (the unstable one).

Consider the second steady state. The coefficient matrix and the eigenvalues in $O_2\left(1,\frac{1}{2}\right)$ have the form

$$A_2=\begin{bmatrix}-2 & -2\\ 2 & 0\end{bmatrix},\ \lambda_{1,2}(A_2)=-1\pm i\sqrt{3}.$$

The equilibrium is a stable focus. The complete phase portrait of the nonlinear system is displayed in Figure 1.3

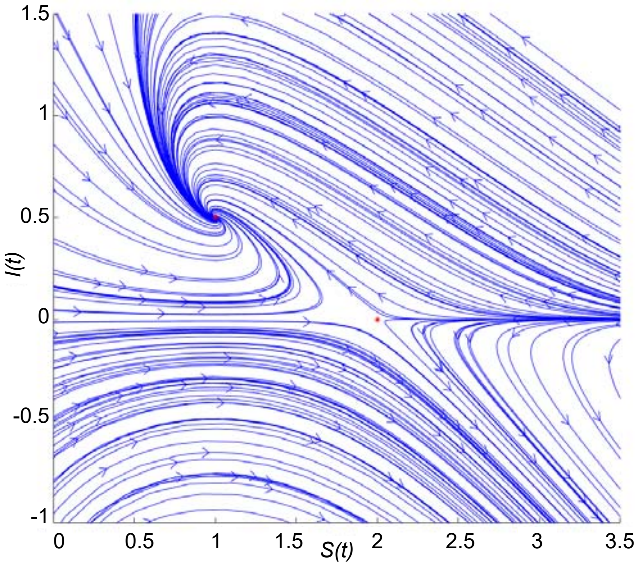

Figure 1.3. Phase portrait of the nonlinear system

2. Epidemiological model with delay

Typically, after interaction with an infectious individual, a susceptible individual does not directly become infectious himself or herself until a certain `incubation' time $\tau > 0$ has elapsed. Thus, any adequate mathematical model is expected to account for this delay effect [17–19]. An SIR-type model incorporating this delay mechanism reads as:

$$\frac{dS(t)}{dt} = \lambda - \mu S(t) - \beta S(t) I(t-\tau),$$
$$\frac{dI(t)}{dt} = \beta S(t-\tau) I(t) - (\mu+\gamma) I(t), \tag{2.1}$$
$$\frac{dR(t)}{dt} = \gamma I(t-\tau) - \mu R(t).$$

2.1. Steady states

Again, determining the steady states amounts to solving the algebraic system

$$\lambda - \mu S - \beta SI = 0,\ \beta SI - (\mu+\gamma) I = 0.$$

Hence, system (2.1) has the same steady states $O_1\left(\frac{\lambda}{\mu}, 0\right)$, $O_2\left(\frac{\mu+\gamma}{\beta}, \frac{\lambda}{\mu+\gamma} - \frac{\mu}{\beta}\right)$ as the instantaneous one, both lying in the first quadrant.

2.2. Steady state investigation

The linear approximation of the system around $O(S_0, I_0)$ is given by

$$\frac{dS(t)}{dt} = [-\mu - \beta I(t-\tau)]_{(S_0,I_0)}(S(t) - S_0) - \beta S(t)\big|_{(S_0,I_0)}(I(t-\tau) - I_0),$$
$$\frac{dI(t)}{dt} = \beta I(t)_{(S_0,I_0)}(S(t-\tau) - S_0) + [\beta S(t-\tau) - (\mu+\gamma)]_{(S_0,I_0)}(I(t) - I_0). \tag{2.2}$$

Note the slight abuse of notation in Equation (2.2): When writing $I(t-\tau)|_{I_0}$, we actually mean $I|_{I_0}$, etc.

Consider each of the steady states $O_1(S_1, I_1)$, $O_2(S_2, I_2)$ separately.

1. Plugging the first steady state $O_1\left(\frac{\lambda}{\mu}, 0\right)$ into the linear approximation Equation (2.2), we get

$$\frac{dS(t)}{dt} = -\mu\left(S(t) - \frac{\lambda}{\mu}\right) - \lambda\frac{\beta}{\mu} I(t-\tau),\ \frac{dI(t)}{dt} = \left(\lambda\frac{\beta}{\mu} - (\mu+\gamma)\right) I(t) \tag{2.3}$$

or, in the matrix form,

$$\frac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = \begin{bmatrix} -\mu & 0 \\ 0 & \lambda\frac{\beta}{\mu} - (\mu+\gamma) \end{bmatrix}\begin{pmatrix} S(t) - \frac{\lambda}{\mu} \\ I(t) \end{pmatrix} + \begin{bmatrix} 0 & -\lambda\frac{\beta}{\mu} \\ 0 & 0 \end{bmatrix}\begin{pmatrix} S(t-\tau) - \frac{\lambda}{\mu} \\ I(t-\tau) \end{pmatrix}. \tag{2.4}$$

Thus, the coefficient matrices A_1 and B_1 read as

$$A_1 = \begin{bmatrix} -\mu_1 & 0 \\ 0 & \lambda\frac{\beta}{\mu} - (\mu+\gamma) \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & -\lambda\frac{\beta}{\mu} \\ 0 & 0 \end{bmatrix},$$

so that the resulting characteristic equation is given by

$$\det\{A_1 + e^{-\xi\tau} - \xi E\} = \begin{vmatrix} -\mu - \xi & -\lambda\frac{\beta}{\mu} e^{-\xi\tau} \\ 0 & \lambda\frac{\beta}{\mu} - (\mu+\gamma) - \xi \end{vmatrix} = (-\mu - \xi) \times \left[\lambda\frac{\beta}{\mu} - (\mu+\gamma) - \xi\right] = 0.$$

The eigenvalues remain unchanged:

$$\xi_1 = -\mu < 0,\ \xi_2 = \lambda\frac{\beta}{\mu} - (\mu+\gamma) > 0.$$

In contrast to planar systems without delay, there is no comparable classification of steady states for systems with delay. In [19, 20], it was shown that the trajectories of the linearized system get eventually 'glued' after a finite number of states so that the steady state remains a saddle. Such systems are called systems with weak delay.

2. Turning to the second steady state $O_2\left(\frac{\mu+\gamma}{\beta}, \frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)$, the linear approximation reads as

$$\frac{dS(t)}{dt} = -\frac{\beta\lambda}{\mu+\gamma}\left(S(t)-\frac{\mu+\gamma}{\beta}\right)-(\mu+\gamma)\left(I(t-\tau)-\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)\right),$$

$$\frac{dI(t)}{dt} = \beta\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right)\left(S(t-\tau)-\frac{\mu+\gamma}{\beta}\right), \tag{2.5}$$

or, in the matrix form,

$$\frac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = \begin{bmatrix} -\frac{\beta\lambda}{\mu+\gamma} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} S(t)-\frac{\mu+\gamma}{\beta} \\ I(t)-\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right) \end{pmatrix} + \begin{bmatrix} 0 & -(\mu+\gamma) \\ \beta\frac{\lambda}{\mu+\gamma}-\mu & 0 \end{bmatrix}\begin{pmatrix} S(t-\tau)-\frac{\mu+\gamma}{\beta} \\ I(t-\tau)-\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right) \end{pmatrix} \tag{2.6}$$

with the coefficient matrices

$$A_2 = \begin{bmatrix} -\frac{\beta\lambda}{\mu+\gamma} & 0 \\ 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & -(\mu+\gamma) \\ \frac{\beta\lambda}{\mu+\gamma}-\mu & 0 \end{bmatrix}$$

and the characteristic equation

$$\det\{A_2+e^{-\xi\tau}B_2-\xi E\} = \begin{vmatrix} -\frac{\beta\lambda}{\mu+\nu}-\xi & -(\mu+\gamma)e^{-\xi\tau} \\ \left(\frac{\beta\lambda}{\mu+\nu}-\mu\right)e^{-\lambda\tau} & -\xi \end{vmatrix} = \xi^2+\frac{\beta\lambda}{\mu+\nu}\xi+[\beta\lambda-\mu(\mu+\gamma)]e^{-2\xi\tau}=0. \tag{2.7}$$

Since no explicit procedures for finding the roots of the characteristic quasi-polynomial (2.7) exist in the general case, Lyapunov's second method [21, 22] will be used for our stability investigation.

It is not difficult to see

$$A_2+B_2 = \begin{bmatrix} -\frac{\beta\lambda}{\mu+\gamma} & -(\mu+\gamma) \\ \frac{\beta\lambda}{\mu+\gamma}-\mu & 0 \end{bmatrix},$$

so that the characteristic equation of A_2+B_2 reads as

$$\det\{A_2+B_2-\xi E\} = \begin{vmatrix} -\frac{\beta\lambda}{\mu+\gamma}-\xi & -(\mu+\gamma) \\ \frac{\beta\lambda}{\mu+\gamma}-\mu & -\xi \end{vmatrix} = \xi^2+\frac{\beta}{\mu+\gamma}\xi+[\beta\lambda-\mu(\mu+\gamma)]=0.$$

Since all coefficients are positive, by virtue of Routh & Hurwitz criterion, the matrix A_2+B_2 is asymptotically stable. Hence, the system linearized around $O_2(S_2, I_2)$ in case of no delay, i.e., for $\tau=0$, is asymptotically stable. As shown before, the equilibrium state is either a stable node (sink) or a stable focus. Let us prove that the steady state $O_2(S_2, I_2)$ of the original nonlinear system with delay is going to be stable as well for 'small' values of $\tau<\tau_0$. To this end, we exploit the Lyapunov's function method with Razumikhin conditions [20, 21].

Introducing the new variables

$$S(t)=S_1(t)-\frac{(\mu+\gamma)}{\beta}, \ I(t)=I_1(t)-\left(\frac{\lambda}{\mu+\gamma}-\frac{\mu}{\beta}\right) \tag{2.8}$$

and substituting them into the first two equations in (2.1), we get

$$\frac{dS_1(t)}{dt} = -\frac{\beta\lambda}{\mu+\gamma}S_1(t)-(\mu+\gamma)I_1(t-\tau)-\beta S_1(t)I_1(t-\tau),$$

$$\frac{dI_1(t)}{dt} = \frac{\beta\lambda-\mu(\mu+\gamma)}{\mu+\gamma}S_1(t-\tau)+\beta S_1(t-\tau)I_1(t). \tag{2.9}$$

Hence, investigating the steady state $O_2(S_2, I_2)$ of the original system (2.1) was reduced to studying the zero equilibrium of (2.9). For our stability investigation, a quadratic Lyapunov function

$$V(S_1, I_1) = h_{11}S_1^2 + 2h_{12}S_1I_1 + h_{22}I_1^2 \tag{2.10}$$

will be used. Since the matrix $A_2 + B_2$ is asymptotically stable, for any positive definite C, the matrix Lyapunov equation

$$(A_2 + B_2)^T H + H(A_2 + B_2) = -2C,\ H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix},\ C = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \tag{2.11}$$

has a positive definite H as the only solution. Reading Equation (2.11) componentwise, due to the symmetry, we get a system of three equations:

$$2\left(-\frac{\beta}{\mu+\gamma}h_{11} + \frac{\beta\lambda}{\mu+\gamma}h_{12}\right) = -2c_{11},\ -(\mu+\gamma)h_{11} - \frac{\beta\lambda}{\mu+\gamma}h_{12} + \frac{\beta\lambda}{\mu+\gamma}h_{22} = -2c_{12},\ -2(\mu+\gamma)h_{12} = -2c_{22}.$$

The unique solution is

$$h_{11} = \frac{\mu+\gamma}{\beta\lambda}\left(c_{11} + \frac{\beta\lambda}{(\mu+\gamma)^2}c_{22}\right),\ h_{12} = \frac{1}{\mu+\gamma}c_{22},$$

$$h_{22} = \frac{\mu+\gamma}{\lambda\beta}\left[\frac{(\mu+\gamma)^2}{\beta\lambda}c_{11} - 2c_{12} + \left(1 + \frac{\beta}{(\mu+\gamma)^2}\right)c_{22}\right]. \tag{2.12}$$

Computing the complete derivative of the Lyapunov function from Equation (2.10) along the solution to the linearized delay system around the second steady state $O_2(S_2, I_2)$, we get

$$\frac{d}{dt}V(S_1(t), I_1(t)) = (S_1(t), I_1(t)C(S_1(t), I_1(t)) \le -\lambda_{\min}(C)[S_1^2(t) + I_2^2(t)].$$

As it was shown in [6], under the condition

$$\tau < \tau_0,\ \tau_0 = \frac{\lambda_{\min}(C)}{2|HB_2|(|A_2| + |B_2|)\sqrt{\sqrt{\phi(H)}}} \text{ and } \phi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)},$$

the zero solution of the system with delay is asymptotically stable implying the second steady state of the original nonlinear equation with delay is steady as well.

Remark. The question of estimating the stability region in the phase space remains open.

Conclusion

We considered an epidemiological model constituted by a system of three ordinary differential equations [14-16]. Its stability investigation was reduced to a qualitative study of a planar system. The steady states were computed and the overall phase portrait was obtained. A modification via introduction of a delay term was obtained [17-19] and the resulting system was also studied.

References:

1. Боев Б. В. Современные этапы математического моделирования процессов развития и распространения инфекционных заболеваний / Б. В. Боев // Эпидемиологическая кибернетика: модели, информация, эксперименты. – М., 1991.
2. Dietz K. Daniel Bernoulli's epidemiological revisited / K. Dietz, J. A. Heesterbeek // Mathematical Biosciences. – 2002. – pp. 1–21.
3. Бароян О. В. Математика и эпидемиология / О. В. Бароян, Л. А. Рвачев. – М. : Знание, 1977.
4. Бейли Н. Математика в биологии и медицине / Н. Бейли. – М. : Мир, 1970.
5. Bailey N. T. J. The Mathematical Theory of Infections Diseases / N. T. J. Bailey. – London, 1975.
6. Черкасский Б. Л. Инфекционные и паразитарные болезни человека / Б. Л. Черкасский. – М. : Медицинская газета, 1994.
7. Енько П. Л. О ходе эпидемий некоторых заразных болезней / П. Л. Енько // Врач. – № 46.
8. Adimy A. Periodic oscillations in leukopoiesis models with two delays / A. Adimy, F. Crauste, S. Ruan // J. Theor. Biol. – 2006. – P. 288–299.
9. DeConde R. Post-transplantation dynamics of the immune response to chronic myelogenous leukemia / R. DeConde, P. S. Kim, D. Levy, P. P. Lee // J. Theor. Biol. – 2005. – pp. 39–59.
10. Hethcote H. W. The mathematics of infectious diseases / H. W. Hethcote // SIAM Review. – 2000.
11. He Zh. L. Nonlinear state-dependent feedback control strategy in the SIR epidemic model with resource limitation / Zh. L. He, J. G.Li, L. F. Nie, Zh. Zhao // J. Adv. in Difference Equations (2017) 2017:209, DOI 10.1186/s13662-017-1229-8.
12. Li C. A study on time-delay rumor propagation model with saturated control function / C. Li // J. Advances in Difference Equations (2017) 2017:255 DOI 10.1186/s13662-017-1319-7.
13. Kang C. Global stability of a diffusive and delayed virus dynamics model with Crowley-Martin incidence function and CTL immune response / C. Kang, H. Miao, X. Chen, J. Xu, D. Huang // J. Adv. in Difference Equations (2017) 2017:324 DOI 10.1186/s13662-017-1332-x.
14. Smith. R. Modelling Disease Ecology with Mathematics / R. Smith // American Institute of Mathematical Sciences. – pp. 1–189.
15. Enatsu Y. Global stability of SIRS epidemic models with a class of nonlinear incidence and distributed delays / Y. Enatsu, Y. Nakata, Y. Muroya // Acta Mathematica Scientia, 32(3): 851–865 (2012)
16. Enatsu Y. Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates / Y. Enatsu, Y. Nakata, G. Izzo, A. Veechio // Journal of Difference Equations and Applications.
17. Nakata Y. Of the Global Stability of an SIRS Epidemic Model with a Distributed Delays / Y. Nakata, Y. Enatsu, Y. Muroya // Discrete and Continuous Dynamical Systems, Supplement 2011. – P. 1119–1128.
18. Хейл Дж. Теория функционально-дифференциальных уравнений / Дж. Хейл. – М. : Мир, 1984. – 421 с.
19. Эльсгольц Л. Э. Введение в теорию дифференциальных уравнений с отклоняющимся аргументом / Л. Э. Эльсгольц, С. Б. Норкин. – М. : Наука, 1970. – 240 с.
20. Бендіткіс Д. Б. Systems with weak delay in the plane / Д. Б. Бендіткіс, Й. Діблік, Д. Я. Хусаінов // Вісн. Київ. ун-ту. Серія "Фізико-математичні науки". – 2001. – Вип. 3. – С. 175–182.

21. Хусаинов Д. Я. Метод функций Ляпунова в исследовании устойчивости дифференциально-функциональных систем / Д. Я. Хусаинов, А. В. Шатырко. – К. : Изд-во Киевского университета, 1997. – 236 с.
22. Хусаінов Д. Я. Стійкість нелінійних систем регулювання з післядією / Д. Я. Хусаінов, А. В. Шатирко. – К. : ДП "Інформ.-аналіт. агентство", 2012. – 73 с.
23. Keeling M. The Mathematics of Diseases, 2001. URL: https://plus.maths.org/content/mathematics-diseases (retrieved October 2017)
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Д. Я. Хусаінов, д-р фіз.-мат. наук, проф.,
А. В. Шатирко, канд. фіз.-мат. наук, доц.
Київський національний університет імені Тараса Шевченка, Київ, Україна,
М. В. Покойовий, д-р природ. наук., молод. проф.
Університет Техасу в Ель-Пасо, факультет математичних наук, Ель-Пасо, Техас, США

АНАЛІЗ ОДНІЄЇ ЕПІДЕМІОЛОГІЧНОЇ МОДЕЛІ З ПІСЛЯДІЄЮ

Досліджено модель процесу захворювання, визначено, які складові моделі впливають на динаміку процесу. Досліджено стійкість стану рівноваги системи без запізнення. Проведено чисельне інтегрування академічного прикладу, побудовано фазовий портрет. Розглянуто систему із запізненням.

Ключові слова: диференціальні рівняння, математична модель, фазовий портрет, особлива точка, система із запізненням.

Д. Я. Хусаинов, д-р физ.-мат. наук, проф.,
А. В. Шатырко, канд. физ.-мат. наук, доц.
Киевский национальный университет имени Тараса Шевченко, Киев, Украина,
М. Покойовый, д-р естест. наук, мл. проф.
Университет Техаса в Эль Пасо, факультет математических наук, Эль-Пасо, Техас, США

АНАЛИЗ ОДНОЙ ЭПИДЕМИОЛОГИЧЕСКОЙ МОДЕЛИ С ПОСЛЕДЕЙСТВИЕМ

Исследована модель процесса заболевания. Выяснено, какие составляющие модели влияют на динамику процесса. Исследована устойчивость положений равновесия системы без запаздывания. Проведено численное интегрирование академического примера и построен фазовый портрет. Рассмотрена система с запаздыванием.

Ключевые слова: дифференциальные уравнения, математическая модель, фазовый портрет, особая точка, система с запаздыванием.

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D. Ya. Khusainov, Dr. Sc., Prof.,
K. M. Fedorova, student
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine,
J. Diblik, Dr. Sc., Prof.,
J. Bastinec, C.Sc., Doc.
Brno University of Technology, Faculty of Electrical Engineering and Communication, Czech Republic, Brno

ABOUT ONE MATHEMATICAL MODEL OF DYNAMICS OF FREE COMPETITION MARKET

The model of free competition market was developed and investigated, found out which components and how affect trade. A program, which was written, allows you to find singular points of system of differential equations and construct a phase portrait. On the example with the given parameters, the model of the market of free competition is constructed and investigated.
Keywords: differential equations, market model, phase portrait, singular point.

Introduction. Modeling of the processes of society's welfare is one of the most actual problems of society at present . One of the most important components is modeling of dynamics of economic growth. And one of the determining components of growth is stability of market relations [5-8]. As a rule, building of mathematical models is based on the balance of products and the possible development of manufacture over a certain time interval. As market, we understand voluntary interaction of buyers and sellers comply with requirements of country[5-8]. As a rule, the mathematical model has equilibrium positions and it is interesting to figure out conditions when this position is asymptotically stable.

Model of free competiton market. Model of dynamics of free competition market could be described by next system of ordinary differential equations [5-8]

$$\frac{dp_j(t)}{dt} = V_j\left(p_j^*, p_j^0, p_j(t)\right) + C_j\left(p_0, p(t)\right) + D_j\left(p_j^{**}, p_j^0, p(t)\right) + G_j\left(q_j^0, q(t)\right). \tag{1}$$

Here $V_j\left(p_j^*, p_j^0, p_j(t)\right)$, $p_j(t) > p_j^*$, $j=\overline{1,n}$ functions, which characterize influencing of seller $D_j\left(p_j^{**}, p_j^0, p(t)\right)$, $p_j(t) < p_j^{**}$, $j=\overline{1,n}$ functions, which characterize influencing of buyer, $C_j\left(p_0, p(t)\right)$, $p(t) = \left(p_1(t), p_2(t), ..., p_n(t)\right)$ functions, which characterize influencing of impact of competition, $G_j\left(q_j^0, q(t)\right)$, $q(t) = \left(q_1(t), q_2(t), ..., q_n(t)\right)$, $j=\overline{1,n}$ functions, which characterize influencing external factors.

The above factors have the biggest influencing on price of goods changing. Some contribute to higher prices, others to lower. Some factors are controlled by the firm, others are not. Nevertheless, together they characterize price of goods changing at a particular moment in time, and this dependence is called free competition.

Entered variables have next meaning:

$p_j(t)$ – the price of j-th good, which is sold at the moment $t>0$, p_j^0 – equilibrium price of j-th good, $q_j(t)$ - number of units of j-th good, which is sold at the moment $t>0$, q_j^0 – equilibrium number of j-th good, p_j^* - the lower threshold value of price j-th good (prime cost), p_j^{**} - the higher threshold value of price j-th good (buyer's possibilities), $p_j' = p_j^0 - p_j^*$ – allowable price difference of j-th good for seller, $p_j'' = p_j^{**} - p_j^0$ – allowable price difference of j-th good for buyer.

In normal trading $p_j(t) > p_j^*$, $j=\overline{1,n}$, that is, the seller should not suffer losses. As functions $V_j\left(p_j^*, p_j^0, p_j(t)\right)$, $C_j\left(p^0, p(t)\right)$, $D_j\left(p_j^{**}, p_j^0, p(t)\right)$, $G_j\left(q_j^0, q(t), p_j^0, p_j(t)\right)$, $j=\overline{1,n}$, где $p^0 = \left(p_1^0, p_2^0, ..., p_n^0\right)$, we could choose functions of fractional-rational form [8]

$$V_j\left(p_j^*, p_j^0, p_j(t)\right) = -v_j p_j' \frac{p_j(t)-p_j^0}{p_j(t)-p_j^*},\ D_j\left(p_j^{**}, p_j^0, p(t)\right) = -d_j p'' \frac{p_j(t)-p_j^0}{p_j^{**}-p_j(t)},$$

$$C_j\left(p^0, p(t)\right) = -\sum_{i=1, i\neq j}^{n} c_{ji} p_j^0 \left(\frac{p_j(t)-p_j^0}{p_j^0} - \frac{p_i(t)-p_i^0}{p_i^0}\right). \quad (2)$$

If we put

$$G_j\left(q_j^0, q(t), p_j^0, p_j(t)\right) = \frac{r_j}{q_j^0}\left[p_j(t) q_j(t) - p_j^0 q_j^0\right]\ q_j(t) = q_j^0\left(1 - e_{jj}\frac{p_j(t)-p_j^0}{p_j^0} + \sum_{i=1, j\neq i}^{n} e_{ji}\frac{p_i(t)-p_i^0}{p_i^0}\right), \quad (3)$$

then we obtain

$$G_j\left(q_j^0, q(t), p_j^0, p_j(t)\right) = r_j\left[p_j(t)\left(1 - e_{jj}\frac{p_j(t)-p_j^0}{p_j^0} + \sum_{i=1, i\neq j}^{n} e_{ji}\frac{p_i(t)-p_i^0}{p_i^0}\right) - p_j^0\right],\ p_j^* < p_j(t) < p_j^{**},$$

$$q_j^* < q_j(t) < q_j^{**},\ j=\overline{1,n}.$$

Recorded system (1) with features (2), (3) it is the system, the right side which is the sum of the linear, quadratic and fractional-rational terms. The system has an equilibrium point (a price that suits for both sides) $p_1(t) = p_1^0$, $p_2(t) = p_2^0$. most interesting thing here is to find conditions when an equilibrium point is asymptotically stable.

1. **System in plane**. In particular, when $j=1,2$, on the plane, we obtain a system of two ordinary differential equations

$$\frac{dp_1(t)}{dt} = -v_1 p_1' \frac{p_1(t)-p_1^0}{p_1(t)-p_1^*} - d_1 p_1'' \frac{p_1(t)-p_1^0}{p_1^{**}-p_1(t)} - c_{12} p_1^0\left(\frac{p_1(t)-p_1^0}{p_1^0} - \frac{p_2(t)-p_2^0}{p_2^0}\right) +$$
$$+ r_1\left[p_1(t)\left(1 - e_1\frac{p_1(t)-p_1^0}{p_1^0} + e_{12}\frac{p_2(t)-p_2^0}{p_2^0}\right) - p_1^0\right],$$

$$\frac{dp_2(t)}{dt} = -v_2 p_2' \frac{p_2(t)-p_2^0}{p_2(t)-p_2^*} - d_2 p_2'' \frac{p_2(t)-p_2^0}{p_2^{**}-p_2(t)} - c_{21} p_2^0\left(\frac{p_2(t)-p_2^0}{p_2^0} - \frac{p_1(t)-p_1^0}{p_1^0}\right) +$$
$$+ r_2\left[p_2(t)\left(1 - e_2\frac{p_2(t)-p_2^0}{p_2^0} + e_{21}\frac{p_1(t)-p_1^0}{p_1^0}\right) - p_2^0\right]. \quad (4)$$

Denote $\left[p_1(t)-p_1^0\right]/p_1^0 = x_1(t)$, $\left[p_2(t)-p_2^0\right]/p_2^0 = x_2(t)$ (replacement "normalization of variables") and rewrite the system in the form of

$$\frac{dx_1(t)}{dt} = x_1(t)\left[-\frac{\nu_1 p_1'}{x_1(t) p_1^0 + p_1'} + \frac{d_1 p_1''}{x_1(t) p_1^0 - p_1''} - c_{12} + r_1(1-e_1)\right] + x_2(t)\left(c_{12} + r_1 e_{12}\right) + r_1\left(e_{12} x_2(t) - e_1 x_1(t)\right) x_1(t),$$

$$\frac{dx_2(t)}{dt} = x_2(t)\left[-\frac{\nu_2 p_2'}{x_2(t) p_2^0 + p_2'} + \frac{d_2 p_2''}{x_2(t) p_2^0 - p_2''} - c_{21} + r_2(1-e_2)\right] + x_1(t)\left(c_{21} + r_2 e_{21}\right) +$$
$$+ r_2\left(e_{21} x_1(t) - e_2 x_2(t)\right) x_2(t). \quad (5)$$

As a result of the replacement, study steady state of equilibrium $p_1(t)=p_1^0$, $p_2(t)=p_2^0$ of system (4) reduces to study of zero position $x_1(t)=0$, $x_2(t)=0$ of system of perturbation equations (5).

We convert system to universal form. Using vector-matrix notations, we could write

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} -c_{12}+r_1(1-e_1) & c_{12}+r_1e_{12} \\ c_{21}+r_2e_{21} & -c_{21}+r_2(1-e_2) \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} +$$

$$+\frac{1}{2}\begin{bmatrix} x_1(t) & x_2(t) & 0 & 0 \\ 0 & 0 & x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} -2e_1r_1 & r_1e_{12} \\ r_1e_{12} & 0 \\ 0 & r_2e_{21} \\ r_2e_{21} & -2e_2r_2 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} -$$

$$-\left(\begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix}\begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix} + \begin{bmatrix} p_1' & 0 \\ 0 & p_2' \end{bmatrix}\right)^{-1} \begin{bmatrix} \nu_1 p_1' & 0 \\ 0 & \nu_2 p_2' \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} +$$

$$+\left(\begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix}\begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix} - \begin{bmatrix} p_1'' & 0 \\ 0 & p_2'' \end{bmatrix}\right)^{-1} \begin{bmatrix} d_1 p_1'' & 0 \\ 0 & d_2 p_2'' \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad (6)$$

We introduce following designations

$$x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ X(t)=\begin{pmatrix} x_1(t) & x_2(t) & 0 & 0 \\ 0 & 0 & x_1(t) & x_2(t) \end{pmatrix},\ X_0(t)=\begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix},\ P_0=\begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix},$$

$$A=\begin{bmatrix} -c_{12}+r_1(1-e_1) & c_{12}+r_1e_{12} \\ c_{21}+r_2e_{21} & -c_{21}+r_2(1-e_2) \end{bmatrix},\ B=\frac{1}{2}\begin{bmatrix} -2e_1r_1 & r_1e_{12} \\ r_1e_{12} & 0 \\ 0 & r_2e_{21} \\ r_2e_{21} & -2e_2r_2 \end{bmatrix},\ P'=\begin{bmatrix} p_1' & 0 \\ 0 & p_2' \end{bmatrix},\ P''=\begin{bmatrix} p_1'' & 0 \\ 0 & p_2'' \end{bmatrix},$$

$$V=\begin{bmatrix} \nu_1 p_1' & 0 \\ 0 & \nu_2 p_2' \end{bmatrix},\ D=\begin{bmatrix} d_1 p_1'' & 0 \\ 0 & d_2 p_2'' \end{bmatrix}.$$

Then system (6) will adopt universal vector-matrix form

$$\frac{d}{dt}x(t)=Ax(t)+X(t)Bx(t)-\left[X_0(t)P_0+P'\right]^{-1}Vx(t)+\left[X_0(t)P_0-P''\right]^{-1}Dx(t). \qquad (7)$$

And study of steady state of equilibrium $p_1(t)=p_1^0$, $p_2(t)=p_2^0$ reduces to study of zero equilibrium position $x_1(t)=0$, $x_2(t)=0$ of systems (5), (7).

2. Analysis of equilibrium position. Investigation zero equilibrium position of systems (5), (7) will be carried out by linearization method. Because when $x_1(t)=-p_1'/p_1^0$, $x_1(t)=p_1''/p_1^0$, $x_2(t)=-p_2'/p_2^0$, $x_2(t)=p_2''/p_2^0$ denominators of right-hand sides have features, then domain of definition of systems (5), (6) is a rectangle

$$D=\left\{(x_1,x_2): -p_1'/p_1^0 < x_1 < p_1''/p_1^0, -p_2'/p_2^0 < x_2 < p_2''/p_2^0\right\}, \qquad (8)$$

and linearization will be held in D.

Equilibrium position determines by system

$$x_1\left[-\frac{\nu_1 p_1'}{x_1p_1^0+p_1'}+\frac{d_1p_1''}{x_1p_1^0-p_1''}-c_{12}+r_1(1-e_1)\right]+x_2(c_{12}+r_1e_{12})+r_1(e_{12}x_2-e_1x_1)x_1=0,$$

$$x_2\left[-\frac{\nu_2 p_2'}{x_2p_2^0+p_2'}+\frac{d_2p_2''}{x_2p_2^0-p_2''}-c_{21}+r_2(1-e_2)\right]+x_1(c_{21}+r_2e_{21})+r_2(e_{21}x_1-e_2x_2)x_2=0. \qquad (9)$$

It is very hard to find analytic solution of this system, especially in general. Аналитически решить систему, в общем виде, сложно. Based on the fact that maximal power of numerators and denominators is two, maximal number of solutions is four. Evident (and the most interesting) solution is $O_1(0,0)$.

Right side of system (7) is the sum of the linear, quadratic, and fractional-rational parts. Linearization of quadratic part in a neighborhood of zero gives zero term and that's why system, linearized to (7), in a neighborhood of zero equilibrium position, has next form

$$\frac{d}{dt}x(t)=\left(A+\tilde{A}\right)x(t),\ \ \tilde{A}=\frac{D}{D(x_1,x_2)}\left\{-\left[X_0(t)P_0+P'\right]^{-1}Vx(t)+\left[X_0(t)P_0-P''\right]^{-1}Dx(t)\right\}\Bigg|_{x(t)\equiv 0}.$$

If put $F(x)=-\left[X_0(t)P_0+P'\right]^{-1}V+\left[X_0(t)P_0-P''\right]^{-1}D$, than we have a system

$$\dot{x}=F(x)x,\ F(x)=\begin{bmatrix} f_{11}(x_1,x_2) & f_{12}(x_1,x_2)\\ f_{21}(x_1,x_2) & f_{22}(x_1,x_2)\end{bmatrix},\ x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

Its Jacobian has form

$$\frac{D(F(x))}{D(x)}=\begin{bmatrix} x_1\frac{\partial}{\partial x_1}f_{11}+f_{11}+x_2\frac{\partial}{\partial x_1}f_{12} & x_1\frac{\partial}{\partial x_2}f_{11}+f_{12}+x_2\frac{\partial}{\partial x_2}f_{12}\\ x_1\frac{\partial}{\partial x_1}f_{21}+f_{21}+x_2\frac{\partial}{\partial x_1}f_{22} & x_1\frac{\partial}{\partial x_2}f_{21}+f_{22}+x_2\frac{\partial}{\partial x_2}f_{22}\end{bmatrix}=$$

$$=\begin{bmatrix} x_1\frac{\partial}{\partial x_1}f_{11}+x_2\frac{\partial}{\partial x_1}f_{12} & x_1\frac{\partial}{\partial x_2}f_{11}+x_2\frac{\partial}{\partial x_2}f_{12}\\ x_1\frac{\partial}{\partial x_1}f_{21}+x_2\frac{\partial}{\partial x_1}f_{22} & x_1\frac{\partial}{\partial x_2}f_{21}+x_2\frac{\partial}{\partial x_2}f_{22}\end{bmatrix}+\begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}.$$

And at point $O(0,0)$ we have

$$\frac{D(F(x))}{D(x)}\Bigg|_{x_1=0,x_2=0}=\begin{bmatrix} f_{11}(0,0) & f_{12}(0,0)\\ f_{21}(0,0) & f_{22}(0,0)\end{bmatrix}=F(0).$$

That's why matrix $\tilde{A}$ has next form

$$\tilde{A}=-(P')^{-1}V-(P'')^{-1}D=-\begin{bmatrix} p_1' & 0\\ 0 & p_2'\end{bmatrix}^{-1}\begin{bmatrix} v_1p_1' & 0\\ 0 & v_2p_2'\end{bmatrix}-\begin{bmatrix} p_1'' & 0\\ 0 & p_2''\end{bmatrix}^{-1}\begin{bmatrix} d_1p_1'' & 0\\ 0 & d_2p_2''\end{bmatrix}==\begin{bmatrix} -v_1-d_1 & 0\\ 0 & -v_2-d_2\end{bmatrix}.$$

Thus, matrix of system of linear approximation at zero point has form

$$A+\tilde{A}=\begin{bmatrix} -c_{12}+r_1(1-e_1)-v_1-d_1 & c_{12}+r_1e_{12}\\ c_{21}+r_2e_{21} & -c_{21}+r_2(1-e_2)-v_2-d_2\end{bmatrix}.$$

Its characteristic equation has form

$$\det(A+\tilde{A}-\lambda E)=\begin{bmatrix} -c_{12}+r_1(1-e_1)-v_1-d_1-\lambda & c_{12}+r_1e_{12}\\ c_{21}+r_2e_{21} & -c_{21}+r_2(1-e_2)-v_2-d_2-\lambda\end{bmatrix}=$$

$$=\lambda^2+\lambda(v_1+d_1+c_{12}-r_1+r_1e_1+v_2+d_2+c_{21}-r_2+r_2e_2)+(v_1+d_1+c_{12}-r_1+r_1e_1)(v_2+d_2+c_{21}-r_2+r_2e_2)-$$
$$-(c_{12}+r_1e_{12})(c_{21}+r_2e_{21}).$$

As follows from stability conditions for systems on plane, when inequalities correct

$$p_1=v_1+d_1+c_{12}-r_1+r_1e_1+v_2+d_2+c_{21}-r_2+r_2e_2>0,$$

$$p_2=(v_1+d_1+c_{12}-r_1+r_1e_1)(v_2+d_2+c_{21}-r_2+r_2e_2)-(c_{12}+r_1e_{12})(c_{21}+r_2e_{21})>0$$

zero equilibrium position will be asymptotically stable.

Let us consider economic interpretation of stability conditions. Influence of competition on changing in the price of goods is not as significant as influence of the state, the seller and the buyer, so coefficients $c_{ij}, i,j=2$ will not be taken into account. If we denote $S_i=r_i(1-e_i)-\nu_i-d_i, i=1,2$, then system has an asymptotically stable zero equilibrium position when $S_1+S_2<0$, $S_1S_2<r_1r_2e_1e_2$.

In other words, system to be stable enough to buyer demand for a particular product, as well as the possibility of the seller to provide the product at a price that will satisfy both sides, was greater than the effect on the formation of prices by the state.

Example 1.1. Let's take the following parameters:

$$v_1=3, d_1=6, c_{12}=-3,\ r_1=0.3, e_1=2,\ e_{12}=-3, p_1^0=8.5,\ p_1'=6.5, p_1''=10,$$

$$v_2=5, d_2=8,\ c_{21}=3, r_2=0.25,\ e_2=2.5, e_{21}=-4,\ p_2^0=6.5, p_2'=3, p_2''=9.$$

Than system has next form

$$\dot{x}_1 = (-\frac{19.5}{8.5x_1+6.5} - \frac{60}{10-8.5x_1} + 2.7)x_1 - 3.9x_2 - 0.6x_1^2 - 0.9x_1x_2,$$

$$\dot{x}_2 = 2x_1 - (\frac{15}{6.5x_2+3} + \frac{72}{9-6.5x_2} + 3.375)x_2 + 0.625x_2^2 - x_1x_2,$$

and corresponding matrices are

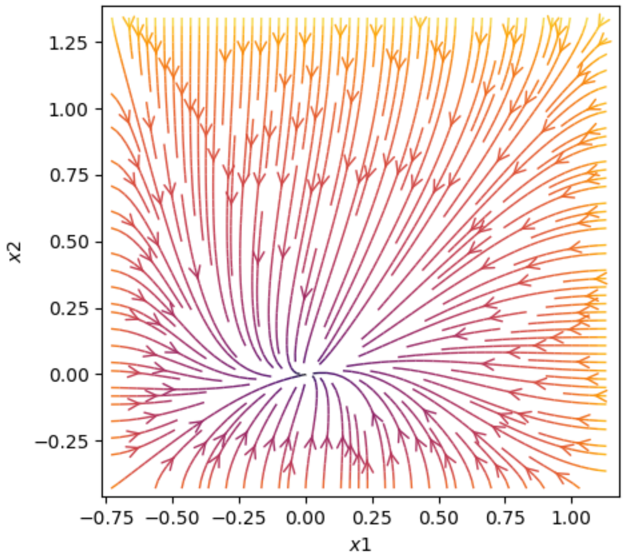
$$P_0 = \begin{bmatrix} 8.5 & 0 \\ 0 & 6.5 \end{bmatrix},\ A = \begin{bmatrix} 2.7 & -3.9 \\ 2 & -3.375 \end{bmatrix},\ B = \begin{bmatrix} -0.6 & -0.45 \\ -0.45 & 0 \\ 0 & -0.5 \\ -0.5 & -0.625 \end{bmatrix},\ P' = \begin{bmatrix} 6.5 & 0 \\ 0 & 3 \end{bmatrix},$$

$$P'' = \begin{bmatrix} 10 & 0 \\ 0 & 9 \end{bmatrix},\ V = \begin{bmatrix} 19.5 & 0 \\ 0 & 15 \end{bmatrix},\ D = \begin{bmatrix} 60 & 0 \\ 0 & 72 \end{bmatrix}.$$

We consider solutions in domain of definition of a system, namely in a rectangle D, where

$$D = \{(x_1, x_2): -13/17 < x_1 < 20/17, -6/13 < x_2 < 18/13\}.$$

Then in this area will be only one special and it is the most interesting point $O_1(0,0)$ from an economic point of view.



Pic. 1

The picture shows that the singular point-point type of node and this equilibrium position is asymptotically stable. The linearized system at (0,0) is a system of two equations

$$\begin{cases} \dot{x}_1 = -6.3x_1 - 3.9x_2, \\ \dot{x}_2 = 2x_1 - 16.375x_2, \end{cases}$$

for which

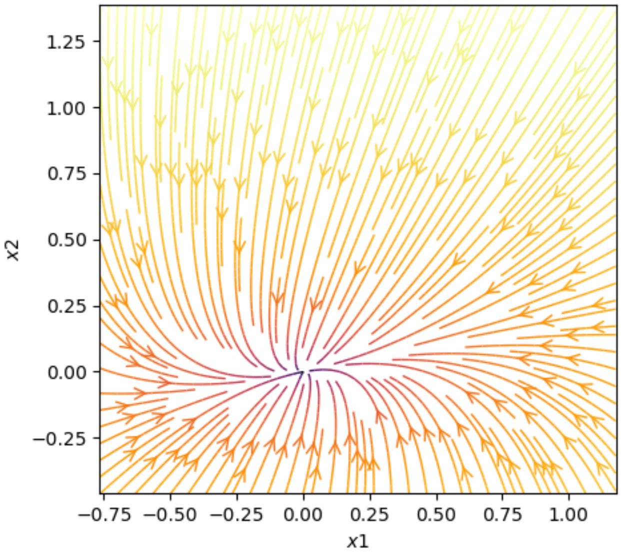
$$A + \tilde{A} = \begin{bmatrix} -6.3 & -3.9 \\ 2 & -16.375 \end{bmatrix}.$$

Its characteristic equation has form

$$\lambda^2 + 22.675\lambda + 110.9625 = 0.$$

Roots of characteristic equation are $\lambda_1 \approx -15.5, \lambda_2 \approx -7.1$, therefore a singular point-stable node.

The phase portrait of the linearized system is constructed in Pic. 2. Comparing Pic. 1 and Pic. 2 it is clear that they are identical, so proposed method of linearization of dynamical system can be used to study the found singular points.



Pic. 2

Comment. The number of singular points is determined by the number of real solutions of system (9). In example 1.1 numerically obtained that in researched area we have only one singular point.

3. System in R^n **space**. Carrying out analogous transformations, we can write the system of differential equations in a general form

$$\frac{dx_1(t)}{dt}=x_1(t)\left[-\frac{\nu_1 p_1'}{x_1(t)p_1^0+p_1'}+\frac{d_1 p_1''}{x_1(t)p_1^0-p_1''}-\sum_{\substack{i=1,\\ i\neq 1}}^{n}c_{1i}+r_1(1-e_1)\right]+\sum_{\substack{i=1,\\ i\neq 1}}^{n}x_i(t)(c_{1i}+r_1e_{1i})+r_1\left(\sum_{\substack{i=1,\\ i\neq 1}}^{n}e_{1i}x_i(t)-e_1x_1(t)\right)x_1(t),$$

$$\frac{dx_2(t)}{dt}=x_2(t)\left[-\frac{\nu_2 p_2'}{x_2(t)p_2^0+p_2'}+\frac{d_2 p_2''}{x_2(t)p_2^0-p_2''}-\sum_{\substack{i=1,\\ i\neq 2}}^{n}c_{2i}+r_2(1-e_2)\right]+\sum_{\substack{i=1,\\ i\neq 2}}^{n}x_i(t)(c_{2i}+r_2e_{2i})+$$

$$+r_2\left(\sum_{\substack{i=1,\\ i\neq 2}}^{n}e_{2i}x_i(t)-e_2x_2(t)\right)x_2(t),$$

....,

$$\frac{dx_n(t)}{dt}=x_n(t)\left[-\frac{\nu_n p_n'}{x_n(t)p_n^0+p_n'}+\frac{d_n p_n''}{x_n(t)p_n^0-p_n''}-\sum_{\substack{i=1,\\ i\neq n}}^{n}c_{ni}+r_n(1-e_n)\right]+\sum_{\substack{i=1,\\ i\neq n}}^{n}x_i(t)(c_{ni}+r_ne_{ni})+$$

$$+r_n\left(\sum_{\substack{i=1,\\ i\neq n}}^{n}e_{ni}x_i(t)-e_nx_n(t)\right)x_n(t).$$

Denoting

$$x(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\\ \cdots\cdots\\ x_n(t)\end{pmatrix},\quad X(t)=\begin{pmatrix}x_1(t) & \dots & x_n(t) & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 0 & x_1(t) & \dots & x_n(t) & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & x_1(t) & \dots & x_n(t)\end{pmatrix},$$

$$X_0(t)=\begin{pmatrix} x_1(t) & 0 & \dots & 0 \\ 0 & x_2(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_n(t) \end{pmatrix},\ P_0=\begin{bmatrix} p_1^0 & 0 & \dots & 0 \\ 0 & p_2^0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n^0 \end{bmatrix},\ P'=\begin{bmatrix} p_1' & 0 & \dots & 0 \\ 0 & p_2' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n' \end{bmatrix},$$

$$P''=\begin{bmatrix} p_1'' & 0 & \dots & 0 \\ 0 & p_2'' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n'' \end{bmatrix},\ V=\begin{bmatrix} \nu_1 p_1' & 0 & \dots & 0 \\ 0 & \nu_2 p_2' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu_n p_n' \end{bmatrix},$$

$$D=\begin{bmatrix} d_1 p_1'' & 0 & \dots & 0 \\ 0 & d_2 p_2'' & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n p_n'' \end{bmatrix},\ A=\begin{bmatrix} -\sum\limits_{\substack{i=1,\\ i\neq 1}}^{n} c_{1i}+r_1(1-e_1) & c_{12}+r_1e_{12} & \dots & c_{1n}+r_1e_{1n} \\ c_{21}+r_2e_{21} & -\sum\limits_{\substack{i=1,\\ i\neq 2}}^{n} c_{2i}+r_2(1-e_2) & \dots & c_{2n}+r_2e_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}+r_ne_{n1} & c_{n2}+r_ne_{n2} & \dots & -\sum\limits_{\substack{i=1,\\ i\neq n}}^{n} c_{ni}+r_n(1-e_n) \end{bmatrix},$$

$$B=\frac{1}{2}\begin{bmatrix} -2e_1r_1 & r_1e_{12} & \dots & r_1e_{1n} \\ r_1e_{12} & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ r_1e_{1n} & 0 & \dots & 0 \\ 0 & r_2e_{21} & \dots & 0 \\ r_2e_{21} & -2e_2r_2 & \dots & r_2e_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & r_2e_{2n} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r_ne_{n1} \\ 0 & 0 & \dots & r_ne_{n2} \\ \dots & \dots & \dots & \dots \\ r_ne_{n1} & r_ne_{n2} & \dots & -2e_nr_n \end{bmatrix},$$

we obtain system in universal vector-matrix form (7).

Consider the system (7) at R^n space.

$$\frac{d}{dt}x(t)=Ax(t)+X(t)Bx(t)-\left[X_0(t)P_0+P'\right]^{-1}Vx(t)+\left[X_0(t)P_0-P''\right]^{-1}Dx(t).$$

As in plane, investigation of the zero equilibrium position of the system (7) we will do with linearization method. The domain of definition of a system (7) has form of a rectangle

$$\tilde{D}_n=\{(x_1,x_2,...,x_n):-p_1'/p_1^0<x_1<p_1''/p_1^0,-p_2'/p_2^0<x_2<p_2''/p_2^0,\ ,...,-p_n'/p_n^0<x_n<p_n''/p_n^0\} \qquad (8)$$

and linearization will be in area $\tilde{D}_n$ in a neighborhood of zero equilibrium position.

Right side of system (7) is the sum of the linear, quadratic, and fractional-rational parts. Linearization of quadratic part in a neighborhood of zero gives zero term and that's why system, linearized to (7), in a neighborhood of zero equilibrium position, has next form

$$\frac{d}{dt}x(t)=\left(A+\tilde{A}\right)x(t),\ \tilde{A}=\frac{D}{D(x_1,x_2,...,x_n)}\left\{-\left[X_0(t)P_0+P'\right]^{-1}Vx(t)+\left[X_0(t)P_0-P''\right]^{-1}Dx(t)\right\}\Bigg|_{x(t)\equiv 0}.$$

If denote

$$F(x) = -\left[X_0(t)P_0 + P'\right]^{-1} V + \left[X_0(t)P_0 - P''\right]^{-1} D,$$

Than we have a system

$$\dot{x} = F(x)x, \; F(x) = \begin{bmatrix} f_{11}(x_1,...,x_n) & f_{12}(x_1,...,x_n) & ... & f_{1n}(x_1,...,x_n) \\ f_{21}(x_1,...,x_n) & f_{22}(x_1,...,x_n) & ... & f_{2n}(x_1,...,x_n) \\ . & . & ... & . \\ f_{n1}(x_1,...,x_n) & f_{n2}(x_1,...,x_n) & ... & f_{nn}(x_1,...,x_n) \end{bmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}.$$

Its Jacobian has form

$$\frac{D(F(x))}{D(x)} = \begin{bmatrix} f_{11} + \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_1} & f_{12} + \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_2} & ... & f_{1n} + \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_n} \\ f_{21} + \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_1} & f_{22} + \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_2} & ... & f_{2n} + \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_n} \\ . & . & ... & . \\ f_{n1} + \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_1} & f_{n2} + \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_2} & ... & f_{nn} + \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_n} \end{bmatrix} =$$

$$= \begin{bmatrix} f_{11} & f_{12} & ... & f_{1n} \\ f_{21} & f_{22} & ... & f_{2n} \\ . & . & ... & . \\ f_{n1} & f_{n2} & ... & f_{nn} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_1} & \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_2} & ... & \sum_{i=1}^{n} x_i \frac{\partial f_{1i}}{\partial x_n} \\ \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_1} & \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_2} & ... & \sum_{i=1}^{n} x_i \frac{\partial f_{2i}}{\partial x_n} \\ . & . & ... & . \\ \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_1} & \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_2} & ... & \sum_{i=1}^{n} x_i \frac{\partial f_{ni}}{\partial x_n} \end{bmatrix}$$

And at point $O(0,0)$ we have

$$\left.\frac{D(F(x))}{D(x)}\right|_{x_1=0,x_2=0,...x_n=0} = \begin{bmatrix} f_{11}(0,0,...,0) & f_{12}(0,0,...,0) & ... & f_{1n}(0,0,...,0) \\ f_{21}(0,0,...,0) & f_{22}(0,0,...,0) & ... & f_{2n}(0,0,...,0) \\ . & . & ... & . \\ f_{n1}(0,0,...,0) & f_{n2}(0,0,...,0) & ... & f_{nn}(0,0,...,0) \end{bmatrix}..$$

That's why matrix $\tilde{A}$ has next form

$$\tilde{A} = -(P')^{-1} V - (P'')^{-1} D = -\begin{bmatrix} p_1' & 0 & ... & 0 \\ 0 & p_2' & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & p_n' \end{bmatrix}^{-1} \begin{bmatrix} v_1 p_1' & 0 & ... & 0 \\ 0 & v' p_2' & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & v' p_n' \end{bmatrix} -$$

$$-\begin{bmatrix} p_1'' & 0 & ... & 0 \\ 0 & p_2''' & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & p_n'' \end{bmatrix}^{-1} \begin{bmatrix} d_1 p_1'' & 0 & ... & 0 \\ 0 & d_2 p_2'' & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & d_2 p_n'' \end{bmatrix} = -\begin{bmatrix} -v_1 - d_1 & 0 & ... & 0 \\ 0 & -v_2 - d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & -v_n - d_n \end{bmatrix}.$$

Thus, matrix of system of linear approximation at zero point, using the notation $S_i = r_i(1-e_i) - \nu_i - d_i, i = 1,...,n$ has a form

$$A+\tilde{A} = \begin{bmatrix} -\sum_{\substack{i=1,\\ i\neq 1}}^{n} c_{1i} + S_1 & c_{12}+r_1e_{12} & \dots & c_{1n}+r_1e_{1n} \\ c_{21}+r_2e_{21} & -\sum_{\substack{i=1,\\ i\neq 2}}^{n} c_{2i} + S_2 & \dots & c_{2n}+r_2e_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}+r_ne_{n1} & c_{n2}+r_ne_{n2} & \dots & -\sum_{\substack{i=1,\\ i\neq n}}^{n} c_{ni} + S_n \end{bmatrix}.$$

Its characteristic equation has form

$$\det(A+\tilde{A}-\lambda E) = \begin{bmatrix} -\sum_{\substack{i=1,\\ i\neq 1}}^{n} c_{1i} + S_1 - \lambda & c_{12}+r_1e_{12} & \dots & c_{1n}+r_1e_{1n} \\ c_{21}+r_2e_{21} & -\sum_{\substack{i=1,\\ i\neq 2}}^{n} c_{2i} + S_2 - \lambda & \dots & c_{2n}+r_2e_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}+r_ne_{n1} & c_{n2}+r_ne_{n2} & \dots & -\sum_{\substack{i=1,\\ i\neq n}}^{n} c_{ni} + S_n - \lambda \end{bmatrix}.$$

Expanding it, in general form, we obtain

$$\lambda^n + d_1\lambda^{n-1} + ... + d_{n-1}\lambda + d_n = 0$$

As is known, necessary condition for stability of zero position of equilibrium is positivity of coefficients, i.e. If zero equilibrium position of original nonlinear system is asymptotically stable, then $d_i > 0, i = \overline{0,n}$.

We examine the last matrix in more detail. Constant coefficients $c_{ij}i, j = \overline{1,n}$ show influense i -th competitor to price j -th good or service. Given value is not significant, proceeding from the fact that the seller's contribution v_j and buyer's d_j to stabilization of market prices, as well as desire to raise taxes, which is characterized by a coefficient r_j have a more significant effect on changing the price of goods. In addition, some coefficients $c_{ij}i, j = \overline{1,n}$, on diagonal of matrix $A+\tilde{A}-\lambda E$ will be reduced to the off-diagonal, because have different signs. Thus, if do not take into account coefficients c_{ij} when the system is tested for stability, then, in order for the system to be stable, it is necessary that $S_i = r_i(1-e_i) - \nu_i - d_i < 0, \ \forall i = \overline{1,n}$.

Conclusion. We always have competition on market, and sellers of similar goods or suppliers of such services always depend on each other. The system of dynamics of free competition market was investigated. The goal was to determine when this system has a stable equilibrium position. In case of markets, equilibrium means a situation when sellers and buyers are collectively satisfied with current combination of prices and sales or purchases, and thus have no incentive to change existing situation. If, for some reason,equilibrium price has not been established, then forces are emerging in market aimed at establishing such price. The ideal economic equilibrium implies the conditions of perfect competition and absence of external effects.

But in a real economy such conditions are not observed: there is no perfect market, there are side effects of entrepreneurial activity, cyclical and structural fluctuations, unemployment, inflation. All of them deduce economy from equilibrium state. However, this does not mean that economic system can not be brought into a state of equilibrium that will correspond to market realities.

The real equilibrium is an equilibrium that is established in economy in conditions of imperfect competition in the presence of external and internal factors of influence on the market. Each of the market participants can influence the trade with a certain coefficient. In the course of the study of the obtained system, it was established that for the equilibrium of the system, the total influence of the seller and the buyer should be more than the influence of the state.

Список використаних джерел:

1. Самуэльсон П. Экономика / П. Самуэльсон, В. Д. Нордхаус. – М. : Бином-КоРус, 1997. – 800 с.
2. Занг В.-Б. Синергетическая экономика. Время и перемены в нелинейной экономической теории / В.-Б. Занг. – М. : Мир, 1999. – 335 с.
3. Лебедев В. В. Математическое моделирование нестационарных экономических процессов / В. В. Лебедев, К. В. Лебедев. – М. : ООО "Тест", 2011. – 336 с.
4. Математична економіка / Т. В. Блудова, И. А. Джалладова, О. І. Макаренко, Г. В. Шуклін. – К. : КНЕУ, 2009. – 464 с.
5. Калитин Б. С. Математические модели первого порядка конкурентного рынка / Б. С. Калитин. – Минск : БГУ, 2012. – 131 с.
6. Калитин Б. С. Устойчивость рынка двух взаимозаменяемых товаров / Б. С. Калитин, Е. В. Трухан // Вестн. Белорус. ун-та. – 2011. – Сер. 1, № 2. – С. 91–95.
7. Калитин Б. С. Динамическая модель производства и реализации продукции / Б. С. Калитин, Т. Б. Сущ // Вестн. Белорус. ун-та. – 2012. – Сер. 1, № 1. – С. 110–116.
8. Калитин Б. Г. Математические модели первого порядка конкурентного рынка / Б. С. Калитин. – Минск : БГУ, 2011. – 131 с.

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Д. Я. Хусаїнов, д-р фіз.-мат. наук, проф.,
К. М. Федорова, студ.
Київський національний університет імені Тараса Шевченка, Київ, Україна,
Й. Діблик, д-р фіз.-мат. наук, проф.,
Я. Баштинець, канд. фіз.-мат. наук, доц.
Технічний університет, факультет електротехніки та зв'язку, Брно, Чеська республіка

ЩОДО ОДНІЄЇ МАТЕМАТИЧНОЇ МОДЕЛІ ДИНАМІКИ РИНКУ ВІЛЬНОЇ КОНКУРЕНЦІЇ

Побудовано й досліджено модель ринку вільної конкуренції, з'ясовано, які складові й у якому обсязі впливають на торгівлю. Написано програму, яка дозволяє знайти особливі точки системи диференціальних рівнянь і побудувати фазовий портрет. На прикладі при заданих параметрах побудовано й досліджено модель ринку вільної конкуренції.

Ключові слова: диференціальні рівняння, модель ринку, фазовий портрет, особлива точка.

Д. Я. Хусаинов, д-р физ.-мат. наук, проф.,
К. М. Федорова, студ.,
Киевский национальный университет имени Тараса Шевченко, Киев, Украина,
Й. Диблик, д-р физ.-мат. наук, проф.,
Я. Баштинец, канд. физ.-мат. наук, доц.
Технический университет, факультет електротехники и связи, Брно, Чешская республика

ОБ ОДНОЙ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ДИНАМИКИ РЫНКА СВОБОДНОЙ КОНКУРЕНЦИИ

Построена и исследована модель рынка свободной конкуренции, выяснено, какие составляющие и в каком объеме влияют на торговлю. Написана программа, которая позволяет найти особые точки системы дифференциальных уравнений и построить фазовый портрет. На примере при заданных параметрах построена и исследована модель рынка свободной конкуренции.

Ключевые слова: дифференциальные уравнения, модель рынка, фазовый портрет, особая точка.

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В І С Н И К

КИЇВСЬКОГО НАЦІОНАЛЬНОГО УНІВЕРСИТЕТУ ІМЕНІ ТАРАСА ШЕВЧЕНКА

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ВПЦ "Київський університет"
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☎ (38044) 239 3222; (38044) 239 3172; тел./факс (38044) 239 3128
e-mail: vpc_div.chief@univ.net.ua; redaktor@univ.net.ua
http: vpc.univ.kiev.ua
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