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This issue of the journal is dedicated to the 70th anniversary of the well-known scientist in the field of numerical mathematics and scientific computing Professor Ivan Gavrilyuk. All authors cordially congratulate the jubilee and wish him good health and new interesting scientific results.

Responsible Editor R. Chapko

IVAN GAVRILYUK – 70

Our friend of long standing, colleague, and collaborator, Professor Ivan Gavrilyuk (german Gawriljuk), has turned 70.

He was born and grew in the village Majdan Stasiv (currently Goncharivka) of Lityns'kyj district of Vinnitsa region, Ukraine. School he graduated in the village Klembivka of Jampil' district of the same region. His professional activity of almost four decades in two countries, Ukraine and Germany, is a splendid example of ceaseless service to the mathematical community and is noted for remarkable scientific achievements in a wide range of topics in the area of theoretical numerical analysis, mathematical modelling, and scientific computing.

I.P. Gavrilyuk studied mechanics and mathematics at the Faculties of Mechanics and Mathematics and then at the Faculty of Cybernetics of the Taras Shevchenko Kiev State University. He graduated in 1971 from the department of Cybernetics and, as a talented young mathematician, was appointed as assistant professor at the department. His mentors, collaborators, and colleagues at that time were G.N. Polozhij, V.M. Glushkov, V.L .Makarov and other well-known mathematicians from the Kiev school. In 1975 he defended his thesis for the degree of Candidate of Sciences in physics and mathematics at the Taras Shevchenko Kiev State University. In 1979 he was promoted to the post of associate professor of applied statistics and soon to associate professor of computational methods in mathematical physics.

In the period from 1981 to 1989 Makarov and Gavrilyuk were, respectively, chair and vice-chair of the Department of numerical methods of mathematical physics at the Kiev National University of Ukraine. Under their leadership the department became a leading organization in Ukraine in the area of numerical and applied mathematics. Makarov and Gavrilyuk were largely responsible for the grown prestige of the department and for the raised quality of research. Dr. Gavrilyuk was part of a team of young scientists with a vigorous research program and close scientific collaboration with the world-renowned mathematical schools.

In 1989 Dr. Gavrilyuk made a cruicial decision to move to Germany with his wife Ingrid and their children Alexander and Kristina. That year was a turning point in the European history, when young professionals were looking for new opportunities in the new world that was about to be created after the fall of the Berlin wall. In 1989–1999 Dr. Gavrilyuk was a Lecturer, Privatdozent at the Institute of Mathematics, Faculty of Mathematics and Informatics, University of Leipzig and in 1995 he defended his Dr. rer. nat. habilitation at this university. His close collaborators and mentors in Leipzig were the well-known mathematicians Eberhard Zeidler, Damir Arov and Wolfgang Hackbush.

In 1999 Dr. Gavrilyuk was appointed Professor and Chairman of the department of Information and Communication Technologies at the newly founded University of Cooperative Education, Berufsakademie Eisenach, Staatliche Studienakademie Thueringen, later transformed into dual University Gera-Eisenach. These universities represent a new internationally recognized education form, so to say the german "know-how" in the field of closed to practice education. Professor Gawriljuk made a significant contribution to the development of this form of education.

In the earlier period of his professional career as scientist, namely 1971–1975, Dr. Gavrilyuk's research was focused on the theory of finite difference schemes. In this period he initiated a study of a new class of finite difference schemes, namely schemes with exact and explicit spectra. He also introduced the concept of the best scheme with exact spectrum, which was the forerunner of the modern spectral and pseudospectral methods. Dr. Gavrilyuk made important contributions to the development of the theory of exact and truncated difference schemes for variational inequalities and for degenerate ODE's, the direction initiated and developed into a powerful numerical tool in the early 1960s by A.N. Tikhonov and A.A. Samarskii and later in the 1970s by V.L. Makarov. Among the most spectacular achievements of Dr. Gavrilyuk in this area are his results on the existence and uniqueness of exact difference schemes for the weak solutions. They have been used further as the basis for the construction of truncated difference schemes of arbitrary given degree of accuracy as well as of difference schemes on a finite grid for ordinary and partial differential equations in unbounded domains. In the period from 1975 to 1989 Dr. Gavrilyuk participated also in a number of theoretical and applied projects related to mathematical modelling and computer-aided design of complex radioengineering systems. He headed a team for developing a mathematical model of photon recycling diode and used it for computer simulation of photon recycling. It was probably the first mathematical model which could completely describe all complex processes in this electronic device. Due to the strong nonlinearity and nonlocal terms the investigation of this model and its discreztization was a challenging mathematical problem. Further, Dr. Gavrilyuk and his team proposed a new model (a system of nonlinear partial differential equations) of internal-diffusion kinetics of adsorption, derived an appropriate discretization, and developed efficient algorithms and computer programs for its numerical solution. This was a team-work of applied mathematicians and engineers that led to a number of unique results in terms of mathematical modelling, development of numerical algorithms and software for computer simulation.

In 1989 Dr. Gavrilyuk, while working at the University of Leipzig, began a new line of research. He studied differential equations with operator coefficients and other operator equations in Hilbert and Banach spaces, which can be considered as meta-models for partial differential equations. Using the Cayley transform and special functions he obtained the solution operators and closed form solutions of these meta-models containing, e.g., all the three important classes of partial differential equations (parabolic, hyperbolic and elliptic), operator equations (including Lyapunov, Silvester, and other important equations). On the basis of these explicit solutions he was able to construct and justify numerical schemes without accuracy saturation and with exponential accuracy.

Further Dr. Gavrilyuk applied the improper Dunford-Cauchy integral to represent the solution operators and to discretize them using Sinc-quadratures. These algorithms have three important properties: a) they converge exponentially, b) they can be parallelized, and c) in the case of multidimensional problems they allow a tensor-product representation. These important properties yield efficient numerical algorithms of optimal or low complexity, which in the case of multidimensional problems solve the famous "curse of dimensionality" problem. The tensor-product representations of the solution operators has become a crucial tool (very often the only working tool) for many multidimensional problems and is intensively developing at various scientific institutions. Dr. Gavrilyuk's colleague, friend and collaborator in this important field from Leipzig school is Boris Khoromskij.

An important field of Dr. Gavrilyuk's scientific activities in University of Leipzig was mathematical modelling of the sloshing of liquids in moving containers in various marine applications. These phenomena are described by a complex system of nonlinear partial differential equations in domains with moving boundaries. The main idea of the approach used by Dr. Gavrilyuk in a team with I. Lukovskyj, V. Makarov, A. Timokha, M. Hermann and others is to derive simpler mathematical models (so-called modal models) in the form of a system of ODEs. Then he proposed efficient numerical algorithms that for various applications lead to boundary-value, initial-value, or eigenvalue problems for the modal models.

Dr. Gavrilyuk has shown how the seemingly "abstract" mathematical results in terms of numerical functional analysis in Hilbert and Banach spaces could be converted into practical algorithms for solving particular applied problems connected with the sloshing of liquids. In fact, using the full arsenal of theoretical mathematical tools for the computational practice is very typical for the research of Dr. Gavrilyuk.

Professor I.P. Gavrilyuk lectured for 18 years at the Kiev University, then for 10 years at the University of Leipzig and afterwards till now at the dual Gera-Eisenach-university. He has given a whole spectrum of undergraduate, graduate, and special topics courses in numerical methods, computer science, and mathematical modelling and has supervised a large number of diplomas and Ph. D. theses.

Results published by Prof. Gavrilyuk are widely known in the scientific world and make an important contribution to mathematics. Scientific achievements of Professor Gavrilyuk were awarded the State Prize of Ukraine in the field of science and technology.

As editor Prof. Gavrilyuk left his mark in a number of mathematical journals, e.g., Mathematics of Computation, Computational Methods in Applied Mathematics, Journal of Numerical and Applied Mathematics. He has been invited speaker at a number of International conferences, symposia, and workshops. Prof. Gavrilyuk is the author or co-author of 9 monographs, a number of university textbooks, and more than 150 research papers.

He is full of energy, new scientific ideas, and research endeavours. We warmly congratulate the jubilee and wish him good health, fulfilment of his plans, and Many Happy Returns of The Day!

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ON THE NON-LINEAR INTEGRAL EQUATION METHOD FOR THE RECONSTRUCTION OF AN INCLUSION IN THE ELASTIC BODY

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Резюме. Для знаходження границі об'єкту в пружній двовимірній області за відомими даними Коші на її границі застосовано метод нелінійних інтегральних рівнянь, що грунтується на пружних потенціалах. Розроблено ітераційний метод для наближеного розв'язування отриманих інтегральних рівнянь. Знайдено похідну Фреше відповідного оператора і показано розв'язність лінеаризованої системи. Повну дискретизацію здійснено методом тригонометричних квадратур. Через некоректність до отриманої системи лінійних рівнянь застосовано метод регуляризації Тіхонова. Чисельні експерименти показують, що пропонований метод дає добру точність реконструкції при економних обчислювальних затратах.

Abstract. We apply the non-linear integral equation approach based on elastic potentials for determining the shape of a bounded object in the elastostatic two-dimensional domain from given Cauchy data on its boundary. The iterative algorithm is developed for the numerical solution of obtained integral equations. We find the Fréchet derivative for the corresponding operator and show unique solviability of the linearized system. Full discretization of the system is realized by a trigonometric quadrature method. Due to the inherited ill-possedness in the system of linear equations we apply the Tikhonov regularization. The numerical results show that the proposed method gives a good accuracy of reconstructions with an economical computational cost.

1. Introduction

The idea to reduce the problem of the boundary reconstruction directly to non-linear equations and to employ a regularized iterative procedure was firstly suggested in [18]. The concept consists in the use of the reciprocity gap approach based on Green's integral theorem. This approach was successfully extended in [9, 13, 16, 18, 20] for the case of the Laplace equation and in [11, 12, 14, 15] for the Helmholtz equation. The other possible way for it is related with the Green's function [6,7,10,20]. This method is applicable for the reconstruction of an inclusion in some canonical domains for which the Green's functions are known. In this paper we would like to use the potential theory to receive a system of non-linear integral equations [5] which is equivalent to an inverse boundary problem for the Navier equation. As motivation for this research we consider the extension of the potential approach to the system of differential equations in elasticity and on the other hand the problem of the

Key words. Double connected elastostatic domain; boundary reconstruction; elastic potentials; boundary integral equations; trigonometric quadrature method; Newton method; Tikhonov regularization.

shape reconstruction in the elastic medium is of interest for the solid mechanics community.

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ_1 and Γ_2 such that Γ_1 is contained in the interior of Γ_2.

The corresponding direct problem is: Given a vector function g on Γ_2 consider the Dirichlet problem for a vector function $u \in C^2(D) \cap C^1(\bar{D})$ satisfying the Navier equation

$$\Delta^* u = 0 \quad \text{in } D \tag{1}$$

and the boundary conditions

$$u = 0 \quad \text{on } \Gamma_1, \tag{2}$$

$$Tu = g \quad \text{on } \Gamma_2. \tag{3}$$

Here $\Delta^* u = \mu \Delta u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u$ and

$$Tu = \lambda \mathrm{div}\, u\, \nu + 2\mu(\nu \cdot \mathrm{grad})u + \mu \mathrm{div}(Qu)Q\nu,$$

where ν is an outward unit normal vector to the boundary and the matrix Q is given by $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Constants μ and λ $(\mu > 0, \lambda > -\mu)$ are called the Lame coefficients, they characterize the physical properties of the material. Note that throughout the paper the function spaces have to be understood as vector valued.

It is well-know that the direct mixed boundary value problem has the unique solution [21, Chapter X, §10].

The inverse problem we are concerned with is: Given the Neumannn data g on Γ_2 and the Dirichlet data

$$u = f \quad \text{on } \Gamma_2, \tag{4}$$

determine the shape of the interior boundary Γ_1.

As opposed to the forward boundary value problem, the inverse problem is nonlinear and ill-posed.

The issue of uniqueness, i.e., identifiability of the unknown curve Γ_1 from the Cauchy data on Γ_2, is settled by the following theorem (see [4]).

Theorem 1. *Let Γ_1 and $\widetilde{\Gamma}_1$ be two closed curves contained in the interior of Γ_2 and denote by u and $\widetilde{u}$ the solutions to the mixed problem (1)–(3) for the interior boundaries Γ_1 and $\widetilde{\Gamma}_1$, respectively. Assume that $g \neq 0$ and*

$$u = \widetilde{u}$$

on an open subset of Γ_2. Then $\Gamma_1 = \widetilde{\Gamma}_1$.

2. Nonlinear integral equations and iterative schemes for its solutions

Firstly we introduce the single-layer elasticity potential. As it is well known, the fundamental solution to the Navier equation (1) is given by

$$\Phi(x, y) = \frac{c_1}{\pi}\,\ln \frac{1}{|x - y|}\, I + \frac{c_2}{\pi}\, J(x - y),$$

where $c_1 = \frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}$, $c_2 = \frac{\lambda+\mu}{4\mu(\lambda+2\mu)}$, I is the identity matrix and the matrix J is defined by

$$J(w) = \frac{w\,w^\top}{|w|^2}$$

in terms of a dyadic product of $w \in \mathbb{R}^2 \setminus \{0\}$ and its transpose $w^\top$. Then the single-layer potential with vector density ψ on Γ_ℓ is defined by

$$(U_\ell\psi)(x) := \int_{\Gamma_\ell} \Phi(x,y)\psi(y)\,ds(y), \quad x \in D, \quad \ell = 1,2.$$

We search the solution of the boundary value problem (1)–(3) in the form

$$u(x) = (U_1\psi_1)(x) + (U_2\psi_2)(x), \quad x \in D. \tag{5}$$

From the boundary behavior properties of the single-layer elasticity potential [21], we obtain

$$u(x) = (S_{\ell 1}\psi_1)(x) + (S_{\ell 2}\psi_2)(x), \quad x \in \Gamma_\ell, \quad \ell = 1,2 \tag{6}$$

and

$$(Tu)(x) = \frac{1}{2}\psi_2(x) + (D_{21}\psi_1)(x) + (D_{22}\psi_2)(x), \qquad x \in \Gamma_2. \tag{7}$$

Here, the boundary integral operators $S_{\ell k}$ and $D_{\ell k}$ are defined by

$$(S_{\ell k}\varphi)(x) = \int_{\Gamma_\ell} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \Gamma_k\ ,$$

$$(D_{\ell k}\varphi)(x) = \int_{\Gamma_\ell} T_x\Phi(x,y)\varphi(y)\,ds(y), \quad x \in \Gamma_k\ .$$

Taking into account the boundary conditions (2) and (3) we receive from (6) a system of integral equations

$$\begin{cases} S_{11}\psi_1 + S_{12}\psi_2 = 0 & \text{on} \quad \Gamma_1, \\ \frac{1}{2}\psi_2 + D_{21}\psi_1 + D_{22}\psi_2 = g & \text{on} \quad \Gamma_2 \end{cases} \tag{8}$$

and the condition (4) leads to the integral equation

$$S_{21}\psi_1 + S_{22}\psi_2 = f \quad \text{on} \quad \Gamma_2. \tag{9}$$

Theorem 2. *The inverse boundary value problem* (1)–(4) *is equivalent to the system of integral equations* (8)–(9).

We will call the equations (8) as the “field” equations and the equation (9) as the “data” equation.

In general, there exist three different iterative methods to solve the system (8)–(9) by linearization:

A. Given initial guess for the boundary Γ_1 and the densities ψ_1 and ψ_2, we linearize all three equations in order to update all the unknowns.
B. Given initial guess for the boundary Γ_1, we solve the subsystem (8) to obtain the densities. Then, keeping the densities fixed we solve the linearized “data” equation (9) to obtain the update for the boundary.

C. Given initial guess for the densities, we solve the linearized “field” equations (8) to obtain Γ_1 and then we solve the linearized “data” equation (9) to obtain the new densities.

The linearization, using Fréchet derivatives of the operators, and the regularization of the ill-posed equations are needed in all methods. However, the iterative method A requires the calculation of the Fréchet derivatives of the operators with respect to all the unknowns and the selection of three regularization parameters at every step. Thus, we prefer to use one of the so-called two-step methods B or C. Between the two methods, the method B is preferable since we solve first a well-posed linear system and then we linearize the “data” equation.

3. Implementation of the two-step method B

3.1. Numerical solution of the “field” integral equations. Assume that boundary curves Γ_1 and Γ_2 have parametric representation

$$\Gamma_\ell = \{x_\ell(t) = (x_{\ell 1}(t), x_{\ell 2}(t))| \quad t \in [0, 2\pi]\}, \quad \ell = 1, 2,$$

where $x_{\ell 1}$, $x_{\ell 2}$ are 2π–periodic and twice continuously differentiable functions.

It gives us the following parametric form for the operator $S_{\ell k}$

$$(S_{\ell k}\psi_k)(x_\ell(t)) = \frac{1}{\pi}\int_0^{2\pi} K_{\ell k}(t,\tau)\psi_k(\tau)d\tau, \quad \ell, k = 1, 2,$$

where $K_{\ell k}(t,\tau) = \pi\Phi(x_\ell(t), x_k(\tau))$ and $\psi_k(t) = \psi(x_k(t))|x_k'(t)|$. Elementary calculations yield the representation of the matrix $K_{\ell\ell}$

$$K_{\ell\ell}(t,\tau) = -\frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I + \tilde{K}_{\ell\ell}(t,\tau), \quad t \neq \tau,$$

where

$$\tilde{K}_{\ell\ell}(t,\tau) = K_{\ell\ell}(t,\tau) + \frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I, \quad t \neq \tau$$

with the diagonal term

$$K_{\ell\ell}(t,t) = \frac{c_1}{2}\ln\left(\frac{1}{e|x_\ell(t)|^2}\right)I + c_2\frac{x_\ell'(t)\cdot x_\ell'(t)^\top}{|x_\ell'(t)|^2}.$$

Parametrization of integral operators $D_{\ell k}$ reads as following

$$(D_{\ell k}\psi_k)(x_\ell(t)) = \frac{1}{\pi}\int_0^{2\pi} L_{\ell k}(t,\tau)\psi_k(\tau)d\tau$$

with the matrices

$$L_{\ell k}(t,\tau) = c_3\frac{(x_\ell(t)-x_k(\tau))\cdot x_\ell'(t)}{|x_\ell'(t)||x_\ell(t)-x_k(\tau)|^2}Q -$$

$$-\frac{(x_\ell(t)-x_k(\tau))\cdot Qx_\ell'(t)}{|x_\ell'(t)||x_\ell(t)-x_k(\tau)|^2}\left\{c_3 I + c_4 J(x_\ell(t)-x_k(\tau))\right\}.$$

Here $c_3 = \dfrac{\mu}{2(\lambda+2\mu)}$ and $c_4 = \dfrac{\lambda+\mu}{\lambda+2\mu}$. The kernels $L_{\ell\ell}$ contain the singularity. The straightforward calculations lead to the following expression

$$L_{\ell\ell}(t,\tau) = \frac{c_3}{2|x'_\ell(t)|}\cot\frac{t-\tau}{2}Q + \tilde{L}_{\ell\ell}(t,\tau),$$

where

$$\tilde{L}_{\ell\ell}(t,\tau) = L_{\ell\ell}(t,\tau) - \frac{c_3}{2|x'_\ell(t)|}\cot\frac{t-\tau}{2}Q$$

with the diagonal term

$$\tilde{L}_{\ell\ell}(t,t) = \frac{c_3 x''_\ell(t)\cdot x'_\ell(t)}{2|x'_\ell(t)|^{3/2}}Q + \frac{x''_\ell(t)\cdot Qx'_\ell(t)}{2|x'_\ell(t)|^{3/2}}\left[c_3 I + c_4\frac{x'_\ell(t)\cdot x'_\ell(t)^\top}{|x'_\ell(t)|^2}\right].$$

Thus we obtain a system of parametrized integral equations

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\left\{\left[-\frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I + \tilde{K}_{11}(t,\tau)\right]\psi_1(\tau)+\right. \\ \qquad\qquad \left. +K_{12}(t,\tau)\psi_2(\tau)\right\}d\tau = 0, \\ \dfrac{\psi_2(t)}{2|x'_2(t)|} + \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\left\{L_{21}(t,\tau)\psi_1(\tau)+\right. \\ \qquad\qquad \left. +\left[\dfrac{c_3}{2|x'_2(t)|}\cot\dfrac{t-\tau}{2}Q + \tilde{L}_{11}(t,\tau)\right]\psi_2(\tau)\right\}d\tau = g(t). \end{cases} \tag{10}$$

For the numerical solution of integral equations (10) we combine a quadrature method and a collocation method based on trigonometric interpolation [3, 17]. For this we choose an equidistant mesh by setting $t_j = jh$, $h = \frac{\pi}{n}$, $j = 0,\dots,2n-1$ and use the following three quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} g(t_k), \tag{11}$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\ln\left(\frac{4}{e}\sin^2 t_j - \frac{\tau}{2}\right)d\tau \approx \sum_{k=0}^{2n-1} R_{|j-k|}\,g(t_k) \tag{12}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\cot\tau - \frac{t_j}{2}d\tau \approx \sum_{k=0}^{2n-1} F_{j-k}\,g(t_k), \tag{13}$$

with the weights

$$R_j = -\frac{1}{2n}\left\{1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos mjh + \frac{(-1)^j}{n}\right\}, \quad F_j = \frac{1}{n}\sum_{m=1}^{n-1}\sin mjh.$$

These interpolation quadrature formulas are obtained by replacing g by its trigonometric interpolation polynomial from the $2n$-dimensional space T_n and then integrating.

Thus we use quadrature rules (11) and (12) to approximate two types of integrals in the integral equations (10) and collocate the approximate equations to obtain the linear system

$$\begin{cases} \displaystyle\sum_{k=0}^{2n-1}\left\{\left[-c_1 R_{|j-k|} I + \frac{1}{n}\,\tilde{K}_{11}(t_j,t_k)\right]\psi_{1n}(t_k) + \right. \\ \qquad\qquad \left. +\dfrac{1}{n}\,K_{12}(t_j,t_k)\psi_{2n}(t_k)\right\} = 0, \\ \\ \dfrac{\psi_{2n}(t_j)}{2|x_2'(t_j)|} + \displaystyle\sum_{k=0}^{2n-1}\left\{\frac{1}{n}L_{21}(t_j,t_k)\psi_{1n}(t_k) + \right. \\ \qquad\qquad \left. + \left[\dfrac{c_3}{|x_2'(t_k)|}F_{j-k}Q + \dfrac{1}{n}\tilde{L}_{22}(t_j,t_k)\right]\psi_{2n}(t_k)\right\} = g(t_j) \end{cases} \tag{14}$$

for $j = 0, 1, \ldots, 2n-1$, which we solve for the nodal values $\psi_{\ell n}(t_k)$, $\ell = 1, 2$ of $\psi_{\ell n} \in T_n$.

The convergence and error analysis for this quadrature method can be established on the basis of the collectively compact operators theory (see [8]) or on the basis of some estimate of trigonometric interpolation in Hölder spaces (see [19]).

Theorem 3. *For $f \in C^{p+1,\beta}[0, 2\pi]$ and a sufficiently large n the system* (14) *has an unique solution with $\psi_{\ell n} \in T_n$ and for the exact solutions ψ_ℓ of* (10) *we have the error estimates*

$$\|\psi_\ell - \psi_{\ell n}\|_{m,\alpha} \le C\frac{\ln n}{n^{p-m+\beta-\alpha}}\|\psi_\ell\|_{p,\beta}, \quad \ell = 1, 2$$

for $0 \le m \le p$, $0 < \alpha \le \beta < 1$ and some constant $C > 0$ depending only on α, β, m, p.

3.2. **Numerical solution of “data” integral equation equation.** According to our algorithm we need to find the correction for Γ_1 from the “data” equation (9), where the densities ψ_ℓ, $\ell = 1, 2$ are known. For simplicity we consider only star-like interior curves, i.e., we choose a parametrization in polar coordinates of the form

$$x_1(t) = \{r(t)c(t) : t \in [0, 2\pi]\}, \tag{15}$$

where $c(t) = (\cos t, \sin t)$ and $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin. Also we use the following notation $S_r\psi = S_{21}\psi$. However, we wish to emphasize that the concepts described below, in principle, are not confined to star-like boundaries only.

For the given r and ψ_ℓ, $\ell = 1, 2$ we solve the linearized ill-posed integral equation

$$(S'[r, \psi_1]q)(t) = f(t) - (S_r\psi_1)(t) - (S_{22}\psi_2)(t) \tag{16}$$

with respect to the function q. Here the Fréchet derivative of the operator S_r has the following representation

$$(S'[r,\psi]q)(t) = \frac{1}{\pi}\int_0^{2\pi} q(\tau)N_r(t,\tau)\psi(\tau)d\tau,$$

where

$$N_r(t,\tau) = -c_1\, c(\tau)\cdot\nabla_{x_1(\tau)} \ln|x_2(t)-x_1(\tau)|I+ \\ + c_2\,(c(\tau),\partial_{x_1(\tau)})J(x_2(t)-x_1(\tau)).$$

Here $(c(\tau),\partial_{x_1(\tau)})J(x_2(t)-x_1(\tau))$ is the tensor obtained by applying $(c(\tau), \partial_{x_1(\tau)})$ to each column of $J(x_2(t)-x_1(\tau))$.

Theorem 4. *The Fréchet derivative operator $S'[r,\widetilde{\psi}_1]$ is injective at the exact solution.*

Proof. Assume $S'[r,\widetilde{\psi}_1]q = 0$. We introduce a function

$$V(x) = \int_{\Gamma_1} (\zeta(y),\partial_y)\Phi(x,y)\psi_1(y)\, ds(y), \qquad x\in \mathbb{R}^2\setminus\Gamma_1,$$

where $\zeta(x_1(t)) = q(t)c(t)$, $t\in[0,2\pi]$.

Clearly the function V satisfies the Navier equation

$$\Delta^* V = 0 \quad \text{in } \mathbb{R}^2\setminus\Gamma_1$$

and by the assumption

$$V^+|_{\Gamma_1} = 0.$$

It is known, [13], that for sufficiently small q, the perturbed interior curve as given in polar coordinates by

$$\Gamma_{1,r+q} = \{(r(t)+q(t))c(t) : t\in[0,2\pi]\}$$

can be represented in terms of the outward unit normal vector ν to $\Gamma_{1,r}$ as follows

$$\Gamma_{1,r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t\in[0,2\pi]\}.$$

Hence, the function V can be rewritten in the form

$$V(x) = \int_0^{2\pi} (\nu(\tau),\partial_{x_1(\tau)})\Phi(x,x_1(\tau))\,\widetilde{q}(\tau)\widetilde{\psi}_1(\tau)\,|x_1'(\tau)|\,d\tau, \qquad x\in\mathbb{R}^2\setminus\Gamma_1.$$

Recalling

$$\Phi(x,y) = \frac{c_1}{\pi}\,\ln\frac{1}{|x-y|}\,I + \frac{c_2}{\pi}\,\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\,\vec{e}_i\otimes\vec{e}_j,$$

and having introduced ε_{ij} the two-dimensional Ricci tensor

$$\tau_i = \varepsilon_{ji}\nu_j, \quad (\varepsilon_{ij}) = Q, \quad \nu = -Q\tau,$$

we rewrite the $(\nu(y),\partial_y)\Phi(x,y)$ in terms of the tangential derivative as follows

$$(\nu(y),\partial_y)\Phi(x,y) = \frac{c_1}{\pi}\,\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,I- \\ -\frac{c_2}{\pi}\,\varepsilon_{ik}\frac{\partial}{\partial\tau(y)}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\,\vec{e}_k\otimes\vec{e}_j$$

By [2, Theorem 4.5] we obtain that the function V can be continuously extended to the boundary Γ_1, i.e.,

$$V(x_1(t))^{\pm} = \mp c_1 \widetilde{\psi}_1(t)\widetilde{q}(t) + \\ + \int_0^{2\pi} (\nu(\tau), \partial_{x_1(\tau)})\Phi(x_1(t), x_1(\tau))\, \widetilde{q}(\tau)\widetilde{\psi}_1(\tau)\, |x_1'(\tau)|\, d\tau.$$

The function V behaves as $o(1)$ at infinity. By the uniqueness of the exterior and interior Dirichlet problem [21, p.55] we have

$$c_1\widetilde{\psi}_1(t)\widetilde{q}(t) = 0, \quad t \in [0, 2\pi].$$

The function u given by (5) solves the Dirichlet problem in the interior of Γ_1. By uniqueness of the solution to the Dirichlet problem for the Navier equation u has to vanish in the interior of Γ_1 and hence $Tu^- = 0$ on Γ_1.

The jump relations imply $Tu^+ = \psi_1$. Employing Holmgren's uniqueness theorem similar to the case for the Helmholtz equation [1, Theorem 2.3.] one can show that the Cauchy data (u^+, Tu^+) cannot be identically zero on an open subset and hence $\widetilde{\psi}_1$ cannot vanish on an open subset of $[0, 2\pi]$. □

For the numerical solution of (16) we apply tha collocation method with the approximation of q in the form

$$q_m = \sum_{i=0}^{2m} q_{mi} l_i, \quad m \in \mathbb{N}, n > m,$$

where $l_i(t) = \cos it$ for $i = 0, \ldots, m$ and $l_i(t) = \sin(m-i)t$ for $i = m+1, \ldots 2m$. Then the following linear system needs to be solved

$$\sum_{j=0}^{2m} q_{mj} A_{ij} = b_i, \quad i = 0, \ldots, 2n-1 \tag{17}$$

with

$$A_{ij} = \frac{1}{n}\sum_{k=0}^{2n-1} l_j(t_k) N_r(t_i, t_k)\psi_{1n}(t_k)$$

and

$$b_i = f(t_i) - \sum_{k=0}^{2n-1} \left\{ \frac{1}{n} K_{21}(t_i, t_k)\psi_{1n}(t_k) + \right. \\ \left. + \left[-c_1 R_{|i-k|} I + \frac{1}{n}\, K_{22}(t_i, t_k) \right] \psi_{2n}(t_k) \right\}.$$

Due to ill-possedness of (17) and its over-determination we apply the least-squares method and the Tikhonov regularization with the regularization parameter $\alpha > 0$.

3.3. **Algorithm for the two-step method B.** Now we summarize the algorithm.

1. Choose some starting value r.
2. Solve the system of well-posed integral equations (8) (see subsec. 3.1).
3. For the given r, ψ_1 and ψ_2 solve the linearized ill-posed integral equation (9) with respect to function q (see subsec. 3.2).
4. Calculate an approximation for the radial function $r = r + \beta q$, where β is a relaxation parameter for the Newton method.
5. Repeat steps 2-4 until a stopping criterion is satisfied.

4. Numerical examples

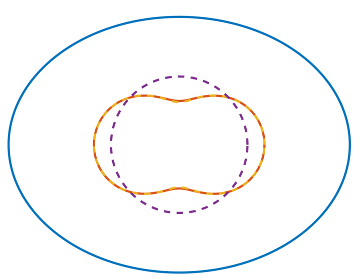
The Cauchy data on Γ_2 were generated by solving the direct problem (1)-(3) for $g = (1,1)^\top$ on Γ_2 and calculating $f = (f_1, f_2)^\top$ as the restriction of the solution on Γ_2. Note that when generating the "exact" Cauchy data we used a finer mesh in order to avoid the "inverse crime". The noisy data were formed as

$$f_\ell^\delta = f_\ell + \delta(2\eta - 1)\|f_\ell\|_{L_2(\Gamma_2)}, \quad \ell = 1, 2$$

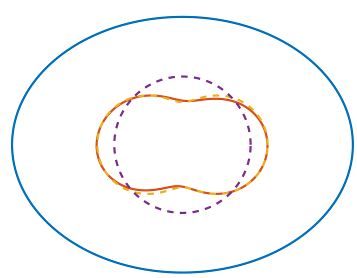
with the noise level δ and the uniformly distributed random variable η in $(0,1)$. The stopping rule was chosen as

$$\frac{\|q\|_{L_2(\Gamma_1)}}{\|r\|_{L_2(\Gamma_1)}} < \epsilon.$$

We demonstrate the feasibility of the proposed methods for the inverse problem (1)-(3) with $\mu = \lambda = 1$ and with following boundaries



a). Reconstruction for exact data after 21 iterations ($\alpha = 1E-10$)



b). Reconstruction for 5% nosy in the data after 16 iterations ($\alpha = 1E-2$)

Fig. 1. Reconstruction of the boundary Γ_1 for Ex. 1

Example 1: The exterior boundary curve Γ_2 is a elipse $\Gamma_2 = \{x_2(t) = (2\cos t, 1.5\sin t), t \in [0, 2\pi]\}$ and the interior boundary curve Γ_1 (to be reconstructed) is peanut-shaped with radial function

$$r(t) = \sqrt{\cos^2 t + 0.25 \sin^2 t}.$$

Example 2: The exterior boundary curve Γ_2 is a rounded rectangle with radial function

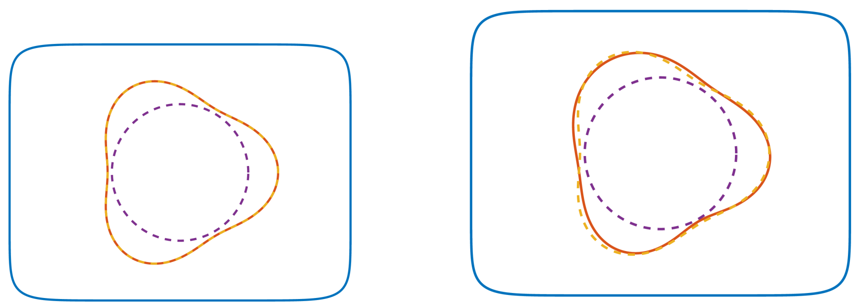
$$r_2(t) = ((1/2\cos t)^{10} + (2/3\sin t)^{10})^{-0.1}$$

and Γ_1 is a boundary with radial function

$$r_1(t) = 1 + 0.15\cos 3t.$$

The results of the numerical experiments for exact and noisy data with $\delta = 5\%$ are reflected on Fig. 1 and Fig. 2. Here we used the following discretization parameters $n = 32$, $m = 4$, $\epsilon = 0.0001$ and $\beta = 0.2$.

Thus, as we see from this preliminary study the non-linear integral equation approach provides accurate reconstruction for exact and noisy data.



a). Reconstruction for exact data after 21 iterations ($\alpha = 1E-10$)

b). Reconstruction for 5% nosy in the data after 20 iterations ($\alpha = 1E-2$)

FIG. 2. Reconstruction of the boundary Γ_1 for Ex. 2

BIBLIOGRAPHY

1. Colton D. Inverse Acoustic and Electromagnetic Scattering Theory (3rd ed.) / D. Colton, R. Kress. – Springer, 2013.
2. Constanda C. Mathematical methods for elastic plates / C. Constanda. – Springer, 2014.
3. Chapko R. On the numerical solution of a boundary value problem in the plane elasticity for a double-connected domain / R. Chapko // Mathematics and Computers in Simulation. – 2004. – Vol. 66. – P. 425-438.
4. Chapko R. On a hybrid method for shape reconstruction of a buried object in an elastostatic half plane / R. Chapko // Inverse Problems and Imaging. – 2009. – Vol. 3. – P. 199-210.
5. Chapko R. The inverse scattering problem by an elastic inclusion / R.Chapko, D. Gintides, L. Mindrinos // Advances in Computational Mathematics. – 2018. – Vol. 44. – P. 453-476.
6. Chapko R. On a nonlinear integral equation approach for the surface reconstruction in semi-infinite-layered domains / R. Chapko, O. Ivanyshyn, O. Protsyuk // Inverse Problems in Science and Engineering. – 2013. – Vol. 21. – P. 547-561.
7. Chapko R.S. On the non-linear integral equation approaches for the boundary reconstruction in double-connected planar domains / R.S. Chapko, O.M. Ivanyshyn Yaman, T.S. Kanafotskyi // Journal of Numerical and Applied Mathematics. – 2016. – Vol. 122. – P. 7-20.

8. Chapko R. On a quadrature method for a logarithmic integral equation of the first kind / R. Chapko, R. Kress // In: *World Scientific Series in Applicable Analysis. – 1993. – Vol. 2. Contributions in Numerical Mathematics* (Agarwal, ed.) – Singapore: World Scientific, P. 127-140.
9. Eckel H. Nonlinear integral equations for the inverse electrical impedance problem / H. Eckel, R. Kress // Inverse Problems. – 2007. – Vol. 23. – P. 475-491.
10. Gavrilyuk I.P. On the numerical solution of linear evolution problems with an integral operator coefficient / I.P. Gavrilyuk, V.L. Makarov, R.S. Chapko // Journal of Integral Equations and Applications. – Spring 1999. – Vol. 11, № 1. – P. 37-56.
11. Ivanyshyn O. Nonlinear integral equation methods for the reconstruction of an acoustically sound-soft obstacle / O. Ivanyshyn, B.T. Johansson // J. Integral Equations Appl. – 2007. – Vol. 19. – P. 289-308
12. Ivanyshyn O. Boundary integral equations for acoustical inverse sound-soft scattering / O. Ivanyshyn, B.T. Johansson // J. Inv. Ill-Posed Problems. – 2008. – Vol. 16. – P. 65-78.
13. Ivanyshyn O. Nonlinear integral equations for solving inverse boundary value problems for inclusions and cracks / O. Ivanyshyn, R. Kress // J. Integral Equations Appl. – 2006. – Vol. 18. – P. 13-38.
14. Ivanyshyn O. Inverse scattering for planar cracks via nonlinear integral equations / O. Ivanyshyn, R. Kress // Mathematical Methods in the Applied Sciences. – 2008. – Vol. 31. – P. 1221-1232.
15. Ivanyshynv O. Identification of sound-soft 3D obstacles from phaseless data / O. Ivanyshynv, R. Kress // Inverse Probl. Imag. – 2010. – Vol. 4. – P. 111-130.
16. Johansson B.T. Reconstruction of an acoustically sound-soft obstacle from one incident field and the far-field pattern / B.T. Johansson, B.D. Sleeman // IMA J. Appl. Math. – 2007. – Vol. 72. – P. 96-112.
17. Kress R. Linear Integral Equations / R. Kress. – Springer, 2014.
18. Kress R. Nonlinear integral equations and the iterative solution for an inverse boundary value problem / R. Kress, W. Rundell // Inverse Problems. – 2005. – Vol. 21. – P. 1207-1223.
19. Kress R. On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation / R. Kress, I.H. Sloan // Numer. Math. – 1993. – Vol. 66. – P. 193-214.
20. Kress R. Iterative methods for planar crack reconstruction in semi-infinite domains / R. Kress, N. Vintonyak // J. Inv. Ill-Posed Problems. – 2008. – Vol. 16. – P. 743-761.
21. Kupradze V.D. Potential methods in the theory of elasticity / V.D. Kupradze. – Jerusalem: Israel Program for Scientific Translations, 1965.

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THE SYSTEM OF POTAPOV'S FUNDAMENTAL MATRIX INEQUALITIES ASSOCIATED WITH A MATRICIAL STIELTJES TYPE POWER MOMENT PROBLEM

B. Fritzsche, B. Kirstein, C. Mädler, M. Scheithauer

Резюме. В статті показано, що множина розв'язків матричної проблеми силових моментів типу Стільєса співпадає з множиною розв'язків системи фундаментальної матриці нерівностей Потапова.

Abstract. The paper shows that the solution set of a matricial Stieltjes-type truncated power moment problem coincides with the solution set of the corresponding system of Potapov's fundamental matrix inequalities.

1. Introduction and preliminaries

The starting point of studying power moment problems on semi-infinite intervals was the famous two part memoir of T. J. Stieltjes [52, 53]. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Krein in collaboration with A. A. Nudelman (see [45, Section 10], [46], [47, Chapter V]). What concerns an operator-theoretic treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [51].

In the 1970's, V. P. Potapov developed a special approach to discuss matrix versions of classical interpolation and moment problems. The main idea of his method is based on transforming such problems into equivalent matrix inequalities with respect to the Löwner semi-ordering. Using this strategy, several matricial interpolation and moment problems could successfully be handled (see, e. g. [6, 7, 13–16, 18, 20–22, 32, 33, 37–44, 48, 54]). L. A. Sakhnovich enriched Potapov's method by unifying the particular instances of Potapov's procedure under the framework of one type of operator identities [9, 35, 50]. Matrix versions of the classical Stieltjes moment problem were studied by Adamyan/Tkachenko [1, 2], Andô [4], Bolotnikov [5, 6, 8], Bolotnikov/Sakhnovich [9], Chen/Hu [11], Chen/Li [12], Dyukarev [17, 18], Dyukarev/Katsnelson [21, 22], and Hu/Chen [34]. The considerations of this paper deal with the more general case of an arbitrary semi-infinite interval $[\alpha, \infty)$, where α is an arbitrarily given real number. This problem has already been treated by other methods in [27, 28].

In order to formulate the concrete moment problem, we are going to study, we first review some notation. Throughout this paper, let p and q be positive

Key words. Stieltjes moment problem; Potapov's fundamental matrix inequalities; Herglotz–Nevanlinna functions; Stieltjes functions.

integers. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of $\upsilon, \omega \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\upsilon,\omega}$ be the set of all integers k for which $\upsilon \leq k \leq \omega$ holds. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p\times 1}$. If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping whose domain is $\mathfrak{A}$ and whose values belong to the set $\mathbb{C}^{q\times q}_{\geq}$ of all non-negative Hermitian complex $q \times q$ matrices is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. By $\mathcal{M}^q_{\geq}(\Omega, \mathfrak{A})$ we denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$. For the integration theory for non-negative Hermitian measures, we refer to [36, 49]. If $\mu = [\mu_{jk}]^q_{j,k=0}$ is a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$ and if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then we use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$ to denote the set of all Borel-measurable functions $f : \Omega \to \mathbb{K}$ for which the integral exists, i. e., that $\int_\Omega |f| \mathrm{d}\tilde{\mu}_{jk} < \infty$ for every choice of j and k in $\mathbb{Z}_{1,q}$, where $\tilde{\mu}_{jk}$ is the variation of the complex measure μ_{jk}. If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$, then let $\int_A f \mathrm{d}\mu := [\int_\Omega 1_A f \mathrm{d}\mu_{jk}]^q_{j,k=1}$ for all $A \in \mathfrak{A}$ and we will also write $\int_A f(\omega)\mu(\mathrm{d}\omega)$ for this integral.

Let $\mathfrak{B}_\mathbb{R}$ (resp. $\mathfrak{B}_\mathbb{C}$) be the σ-algebra of all Borel subsets of $\mathbb{R}$ (resp. $\mathbb{C}$). For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borel subsets of Ω, let $\mathcal{M}^q_{\geq}(\Omega) := \mathcal{M}^q_{\geq}(\Omega, \mathfrak{B}_\Omega)$ and, for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}^q_{\geq,\kappa}(\Omega)$ be the set of all $\sigma \in \mathcal{M}^q_{\geq}(\Omega)$ such that for all $j \in \mathbb{Z}_{0,\kappa}$ the function $f_j : \Omega \to \mathbb{C}$ defined by $f_j(t) := t^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if $\sigma \in \mathcal{M}^q_{\geq,\kappa}(\Omega)$, then we set

$$s^{[\sigma]}_j := \int_\Omega t^j \sigma(\mathrm{d}t) \qquad \text{for each } j \in \mathbb{Z}_{0,\kappa}. \tag{1}$$

The following matricial power moment problem lies in the background of our considerations:

Problem $\mathsf{MP}[\Omega; (s_j)^m_{j=0}, \leq]$**:** Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $(s_j)^m_{j=0}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}^q_{\geq}[\Omega; (s_j)^m_{j=0}, \leq]$ of all $\sigma \in \mathcal{M}^q_{\geq,m}(\Omega)$ for which the matrix $s_m - s^{[\sigma]}_m$ is non-negative Hermitian and for which, in the case $m > 0$, moreover $s^{[\sigma]}_j = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,m-1}$.

The considerations of this paper are mostly concentrated on the case that the set Ω is a one-sided bounded and closed infinite interval of the real axis. Such moment problems are called to be of Stieltjes type. We are going to follow Potapov's strategy to solve the moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)^m_{j=0}, \leq]$, where α is an arbitrarily given real number. After the reformulation of the moment problem in the language of the members of a class of distinguished matrix-valued functions, a first step consists of finding a convenient system of matrix inequalities such that the solution set of the moment problem coincides with the solution set of the system of matrix inequalities. In a second step, one proves a parametrization of the solution set of the system of matrix inequalities, where the case that m is an even integer and the case that m is an odd integer are treated seperately. This paper is aimed at doing the fist step. We are going to construct the system of matrix inequalities in question. It will turn out that the solution set of the moment problem (obtained via Stieltjes transformation) coincides with the solution set of a certain system of Potapov's fundamental matrix inequalities. Further considerations to solve these inequalities will be stated in a subsequent paper.

In Section 2, we recall necessary and sufficient conditions of solvability of the moment problems in question. In Section 3, we give a reformulation of the moment problem, using certain matrix-valued functions. Section 4 is aimed at showing that every solution of the moment problem fulfills necessarily the corresponding system of Potapov's fundamental matrix inequalities. Some integral estimates for the scalar case are given in Section 5. In Section 6, we will prove that each solution of the system of Potapov's fundamental matrix inequalities is a solution of the moment problem as well.

At the end of this section, let us now introduce some further notations, which are useful for our considerations. We will write I_q for the identity matrix in $\mathbb{C}^{q\times q}$, whereas $0_{p\times q}$ is the null matrix belonging to $\mathbb{C}^{p\times q}$. If the size of the identity matrix or the null matrix is obvious, then we will also omit the indexes. The notations $\mathbb{C}^{q\times q}_{\mathrm{H}}$ and $\mathbb{C}^{q\times q}_{\geq}$ stand for the set of all Hermitian complex $q\times q$ matrices and the set of all non-negative Hermitian complex matrices, respectively. If A and B are complex $q\times q$ matrices, then we will write $A\leq B$ or $B\geq A$ to indicate that A and B are Hermitian matrices such that the matrix $B-A$ is non-negative Hermitian. For each $A\in\mathbb{C}^{p\times q}$, let $\mathcal{N}(A)$ be the null space of A and let $\mathcal{R}(A)$ be the column space of A. For each $A\in\mathbb{C}^{q\times q}$, we will use $\Re A$ and $\Im A$ to denote the real part of A and the imaginary part of A, respectively: $\Re A:=\frac{1}{2}(A+A^*)$ and $\Im A:=\frac{1}{2\mathrm{i}}(A-A^*)$. Furthermore, for each $A\in\mathbb{C}^{p\times q}$, let $\|A\|_{\mathrm{F}}$ be the Frobenius norm of A and let $\|A\|_{\mathrm{S}}$ be the operator norm of A. For each $x\in\mathbb{C}^q$, we write $\|x\|_{\mathrm{E}}$ for the Euclidean norm of x. If $n\in\mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $x_j\in\mathbb{C}^{p_j\times q}$ for each $j\in\mathbb{Z}_{1,n}$, then let $\operatorname{col}(x_j)_{j=1}^n:=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$. If $n\in\mathbb{N}$, if $(q_k)_{k=1}^n$ is a sequence of positive integers, and if $y_k\in\mathbb{C}^{p\times q_k}$ for each $k\in\mathbb{Z}_{1,n}$, then let $\operatorname{row}(y_k)_{k=1}^n:=[y_1,y_2,\dots,y_n]$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are non-empty sets with $\mathcal{Z}\subseteq\mathcal{X}$ and if $f\colon\mathcal{X}\to\mathcal{Y}$ is a mapping, then $\operatorname{Rstr}_{\mathcal{Z}} f$ stands for the restriction of f onto $\mathcal{Z}$. Furthermore, let $\Pi_+:=\{z\in\mathbb{C}\colon \Im z\in(0,\infty)\}$ and let $\Pi_-:=\{z\in\mathbb{C}\colon \Im z\in(-\infty,0)\}$.

2. On the solvability of matricial power moment problems

In this section, we recall a necessary and sufficient condition for the solvability of the Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$, where α is an arbitrarily given real number and where m is an arbitrarily given non-negative integer. First we introduce certain sets of sequences of complex $q\times q$ matrices, which are determined by the properties of particular block Hankel matrices built of them. For each $n\in\mathbb{N}_0$, let $\mathcal{H}^{\geq}_{q,2n}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q\times q$ matrices such that the block Hankel matrix $H_n:=[s_{j+k}]_{j,k=0}^n$ is non-negative Hermitian. Furthermore, let $\mathcal{H}^{\geq}_{q,\infty}$ be the set of all sequences

$(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$. The elements of the set $\mathcal{H}_{q,2\kappa}^{\geq}$, where $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, are called *Hankel non-negative definite* sequences. For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there are matrices $s_{2n+1} \in \mathbb{C}^{q \times q}$ and $s_{2n+2} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ to denote the set of sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there is some $s_{2n+2} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. For all $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ are called *Hankel non-negative definite extendable* sequences. For technical reasons, we set $\mathcal{H}_{q,\infty}^{\geq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\geq}$. Observe that the solvability of the matricial Hamburger moment problems can be characterized by the introduced classes of sequences of complex $q \times q$ matrices:

Theorem 2.1 (see, e. g. [10, Theorem 3.2] or [20, Theorem 4.16])**.** *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then*

$$\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset$$

if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$.

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let the sequence $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ be defined by

$$s_{\alpha \triangleright j} := -\alpha s_j + s_{j+1} \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa-1}. \tag{2}$$

The sequence $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ is called the *sequence generated from $(s_j)_{j=0}^{\kappa}$ by right-sided α-shifting*. (An analogous left-sided version is discussed in [25, Definition 2.1].) The sequence $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ is used to define further sets of sequences of complex matrices, which are useful to discuss the Stieltjes moment problems we consider. Let $\mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq}$. For every choice of $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\geq} := \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon (s_{\alpha \triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}\}$. For all $m \in \mathbb{N}_0$, by $\mathfrak{S}_m(\mathbb{C}^{q \times q})$ we denote the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices. Then we set $\mathcal{K}_{q,2n+1,\alpha}^{\geq} := \{(s_j)_{j=0}^{2n+1} \in \mathfrak{S}_{2n+1}(\mathbb{C}^{q \times q}) \colon \{(s_j)_{j=0}^{2n}, (s_{\alpha \triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq}\}$. For all $m \in \mathbb{N}_0$, let $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{m+1} such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. We have $\mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}} = \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon (s_{\alpha \triangleright j})_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}\}$ for all $n \in \mathbb{N}$ and $\mathcal{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}} = \{(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}} \colon (s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}\}$ for all $n \in \mathbb{N}_0$. Obviously, $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}} \subseteq \mathcal{K}_{q,m,\alpha}^{\geq}$. Furthermore, if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$), then we easily see that $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq}$ (resp. $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq,\mathrm{e}}$) holds true for all $\ell \in \mathbb{Z}_{0,m}$. Thus, for all $\alpha \in \mathbb{R}$, let $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{m}$ belongs to $\mathcal{K}_{q,m,\alpha}^{\geq}$ for all $m \in \mathbb{N}_0$, and let $\mathcal{K}_{q,\infty,\alpha}^{\geq,\mathrm{e}} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we call a sequence $(s_j)_{j=0}^{\kappa}$ $[\alpha,\infty)$-*Stieltjes right-sided non-negative definite* (resp. $[\alpha,\infty)$-*Stieltjes* *right-sided non-negative definite expendable*) if it belongs to $\mathcal{K}^{\geq}_{q,\kappa,\alpha}$ (resp. to $\mathcal{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$). Note that left versions of these notions are used in [25, Definition 1.3].

Using the introduced sets of sequences of complex $q \times q$ matrices, we are able to recall solvability criterions of the problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$:

Theorem 2.2 ([19, Theorem 1.4]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$.*

For the description of the solution set $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$ of Problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$, it is essential that one can suppose extendable data without loss of generality:

Theorem 2.3 ([19, Theorem 5.2]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$. Then there is a unique sequence $(\tilde{s}_j)_{j=0}^{m} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ such that the sets $\mathcal{M}^q_{\geq}[[\alpha,\infty);(\tilde{s}_j)_{j=0}^{m},\leq]$ and $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$ coincide.*

3. Some classes of holomorphic matrix-valued functions

The class $\mathcal{R}_q(\Pi_+)$ of all $q \times q$ Herglotz–Nevanlinna functions in the upper half-plane Π_+ consists of all matrix-valued functions $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in Π_+ and which satisfy $\Im[F(\Pi_+)] \subseteq \mathbb{C}^{q\times q}_{\geq}$. Detailed considerations of matrix-valued Herglotz–Nevanlinna functions can be found in [26, 31]. In particular, the functions belonging to $\mathcal{R}_q(\Pi_+)$ admit a well-known integral representation:

Theorem 3.1. (a) *For each $F \in \mathcal{R}_q(\Pi_+)$, there exist unique matrices $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $B \in \mathbb{C}^{q\times q}_{\geq}$ and a unique non-negative Hermitian measure $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that*

$$F(z) = A + zB + \int_{\mathbb{R}} \frac{1+tz}{t-z}\nu(\mathrm{d}t) \qquad \text{for each } z \in \Pi_+. \tag{3}$$

(b) *If $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$, if $B \in \mathbb{C}^{q\times q}_{\geq}$, and if $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$, then $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (3) belongs to $\mathcal{R}_q(\Pi_+)$.*

For each $F \in \mathcal{R}_q(\Pi_+)$, the unique triple $(A,B,\nu) \in \mathbb{C}^{q\times q}_{\mathrm{H}} \times \mathbb{C}^{q\times q}_{\geq} \times \mathcal{M}^q_{\geq}(\mathbb{R})$ for which the representation (3) holds true is called the *Nevanlinna parametrization of F* and we will also write (A_F,B_F,ν_F) for (A,B,ν). In particular, ν_F is said to be the *Nevanlinna measure of F*. If F belongs to $\mathcal{R}_1(\Pi_+)$, then $\mu_F\colon \mathfrak{B}_{\mathbb{R}} \to [0,\infty]$ defined by

$$\mu_F(B) := \int_B (1+t^2)\nu_F(\mathrm{d}t) \qquad \text{for all } B \in \mathfrak{B}_{\mathbb{R}} \tag{4}$$

is a measure, which is called the *spectral measure of F*. By $\mathcal{R}'_q(\Pi_+)$ we denote the set of all $F \in \mathcal{R}_q(\Pi_+)$ for which $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(t) := 1+t^2$ belongs to $\mathcal{L}^1(\mathbb{R},\mathfrak{B}_{\mathbb{R}},\nu_F;\mathbb{R})$. Obviously, $\mathcal{R}'_q(\Pi_+) = \{F \in \mathcal{R}_q(\Pi_+)\colon \nu_F \in \mathcal{M}^q_{\geq,2}(\mathbb{R})\}$. If F belongs to $\mathcal{R}'_q(\Pi_+)$, then $\mu_F\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}_{\geq}$ given by (4) is a well-defined non-negative Hermitian $q \times q$ measure belonging to $\mathcal{M}^q_{\geq}(\mathbb{R})$, which is said to be the *matricial spectral measure of* F. Obviously, for functions which belong to $\mathcal{R}_1'(\Pi_+)$, the notions spectral measure and matricial spectral measure coincide. For our considerations, the class $\mathcal{R}_{0,q}'(\Pi_+)$ of all $F \in \mathcal{R}_q(\Pi_+)$ for which

$$\sup_{y\in[1,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}} < \infty \tag{5}$$

holds true plays an essential role. The class $\mathcal{R}_{0,q}'(\Pi_+)$ is a subclass of $\mathcal{R}_q'(\Pi_+)$ (see, e. g. [26, Lemma 6.1]). The functions belonging to $\mathcal{R}_{0,q}'(\Pi_+)$ admit a particular integral representation:

Theorem 3.2. (a) *For each* $F \in \mathcal{R}_{0,q}'(\Pi_+)$, *there is a unique* $\mu \in \mathcal{M}_{\geq}^q(\mathbb{R})$ *such that*

$$F(z) = \int_{\mathbb{R}} \frac{1}{t-z}\mu(\mathrm{d}t) \qquad \text{for each } z \in \Pi_+, \tag{6}$$

namely the matricial spectral measure of F, *and*

$$\mu(\mathbb{R}) = \lim_{y\to\infty}(y\Im[F(\mathrm{i}y)]) = -\mathrm{i}\lim_{y\to\infty}[yF(\mathrm{i}y)] = \mathrm{i}\lim_{y\to\infty}[yF^*(\mathrm{i}y)].$$

(b) *If* $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ *is a matrix-valued function for which there exists a non-negative Hermitian measure* $\mu \in \mathcal{M}_{\geq}^q(\mathbb{R})$ *such that* (6) *holds true, then* F *belongs to* $\mathcal{R}_{0,q}'(\Pi_+)$.

A proof of Theorem 3.2 is given, e. g., in [14, Theorem 8.7]. If $F \in \mathcal{R}_{0,q}'(\Pi_+)$, then the unique $\mu \in \mathcal{M}_{\geq}^q(\mathbb{R})$ for which (6) holds true is also called the *Stieltjes measure of* F. If a non-negative Hermitian $q\times q$ measure $\mu \in \mathcal{M}_{\geq}^q(\mathbb{R})$ is given, then $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (6) is said to be the *Stieltjes transform of* μ.

Lemma 3.3. *Let* $M \in \mathbb{C}^{q\times q}$ *and let* $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ *be a matrix-valued function which is holomorphic in* Π_+ *and which satisfies the inequality*

$$\begin{bmatrix} M & F(z) \\ F^*(z) & \frac{F(z)-F^*(z)}{z-\overline{z}} \end{bmatrix} \geq 0$$

for each $z \in \Pi_+$. *Then* F *belongs to* $\mathcal{R}_{0,q}'(\Pi_+)$ *and the inequality*

$$\sup_{y\in(0,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}} \leq \|M\|_{\mathrm{S}}$$

holds true. Furthermore, the Stieltjes measure μ *of* F *fulfills* $\mu(\mathbb{R}) \leq M$.

A proof of Lemma 3.3 is given, e. g., in [14, Lemma 8.9].

In view of the Stieltjes moment problem, a further class of matrix-valued functions plays a key role: For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{q;[\alpha,\infty)}$ be the set of all matrix-valued functions $S\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$ and which satisfy $\Im[S(\Pi_+)] \subseteq \mathbb{C}_{\geq}^{q\times q}$ as well as $S((-\infty,\alpha)) \subseteq \mathbb{C}_{\geq}^{q\times q}$. In [29, Theorems 3.1 and 3.6, Proposition 2.16], integral representations of functions belonging to $\mathcal{S}_{q;[\alpha,\infty)}$ are proved. Furthermore, several characterizations of the class $\mathcal{S}_{q;[\alpha,\infty)}$ are given in [29, Section 4]. For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{0,q;[\alpha,\infty)}$ be the class of all $F \in \mathcal{S}_{q;[\alpha,\infty)}$ which satisfy (5). The functions belonging to $\mathcal{S}_{0,q;[\alpha,\infty)}$ admit a particular integral representation. Before we state this, let us note the following:

Remark 3.4. *For every choice of $\alpha \in \mathbb{R}$ and $z \in \mathbb{C} \setminus [\alpha, \infty)$, the function $b_{\alpha,z} \colon [\alpha, \infty) \to \mathbb{C}$ given by $b_{\alpha,z}(t) := 1/(t - z)$ is a bounded and continuous function which, in particular, belongs to $\mathcal{L}^1([\alpha, \infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ for all $\sigma \in \mathcal{M}^q_{\geq}([\alpha, \infty))$.*

Theorem 3.5 ([29, Theorem 5.1]). *Let $\alpha \in \mathbb{R}$.*

(a) *If $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$, then there is a unique $\sigma \in \mathcal{M}^q_{\geq}([\alpha, \infty))$ such that*

$$S(z) = \int_{[\alpha,\infty)} \frac{1}{t - z} \sigma(\mathrm{d}t) \qquad \text{for each } z \in \mathbb{C} \setminus [\alpha, \infty). \tag{7}$$

(b) *If $\sigma \in \mathcal{M}^q_{\geq}([\alpha, \infty))$ is such that $S \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ can be represented via (7), then S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$.*

If $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is given, then the unique $\sigma \in \mathcal{M}^q_{\geq}([\alpha, \infty))$ which fulfills the representation (7) of F is called the $[\alpha, \infty)$-*Stieltjes measure of* F. If $\sigma \in \mathcal{M}^q_{\geq}([\alpha, \infty))$ is given, then $F \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ defined by (7) is said to be the $[\alpha, \infty)$-*Stieltjes transform of* σ. In view of Theorem 3.5, the moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)^m_{j=0}, \leq]$ admits a reformulation in the language of $[\alpha, \infty)$-Stieltjes transforms:

Problem $\mathsf{S}[[\alpha, \infty); (s_j)^m_{j=0}, \leq]$: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)^m_{j=0}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)^m_{j=0}, \leq]$ of all $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$ the $[\alpha, \infty)$-Stieltjes measure of which belongs to

$$\mathcal{M}^q_{\geq}[[\alpha, \infty); (s_j)^m_{j=0}, \leq].$$

Remark 3.6. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$. Then $F_\square := \mathrm{Rstr}_{\Pi_+} F$ belongs to $\mathcal{R}'_{0,q}(\Pi_+)$, the matricial spectral measure $\mu_\square$ of $F_\square$ fulfills $\mu_\square((-\infty, \alpha)) = 0$, and $\sigma := \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu_\square$ is exactly the $[\alpha, \infty)$-Stieltjes measure of F (see [29, Proposition 2.16]).*

4. From the Stieltjes moment problem to the system of Potapov's fundamental inequalities

In this section, we introduce the system of Potapov's fundamental matrices corresponding to the matricial Stieltjes moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)^m_{j=0}, \leq]$. We will see that each solution of this moment problem fulfills necessarily the system of Potapov's fundamental matrix inequalities. First it seems to be useful to introduce further notations and, in particular, several block Hankel matrices which will play a key role in our considerations. For technical reason, let $s_{-1} := 0_{p\times q}$.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)^\kappa_{j=0}$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $H_n := [s_{j+k}]^n_{j,k=0}$, for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $K_n := [s_{j+k+1}]^n_{j,k=0}$, and, for each $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$, let $G_n := [s_{j+k+2}]^n_{j,k=0}$. If m and n are integers such that $-1 \leq m \leq n \leq \kappa$, then we set $y_{m,n} := \mathrm{col}(s_j)^n_{j=m}$ and $z_{m,n} := \mathrm{row}(s_k)^n_{k=m}$. Let $u_0 := 0_{p\times q}$, $\mathfrak{u}_0 := 0_{p\times q}$, $w_0 := 0_{p\times q}$, and $\mathfrak{w}_0 := 0_{p\times q}$. For all $n \in \mathbb{N}$ with $n \leq \kappa + 1$, let $u_n := -y_{-1,n-1}$, and $w_n := z_{-1,n-1}$. Further, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathfrak{u}_n := \left[\begin{smallmatrix} -y_{n+1,2n} \\ 0_{p\times q} \end{smallmatrix}\right]$

and $\mathfrak{w}_n := [z_{n+1,2n}, 0_{p\times q}]$. If a real number α is additionally given, then we continue to use the notation given by (2), and we set $H_{\alpha\triangleright n} := [s_{\alpha\triangleright j+k}]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

For each $n \in \mathbb{N}_0$, we set

$$T_{q,n} := [\delta_{j,k+1} I_q]_{j,k=0}^{n}, \quad v_{q,n} := \operatorname{col}(\delta_{j,0} I_q)_{j=0}^{n}, \quad \text{and} \quad \mathfrak{v}_{q,n} := \operatorname{col}(\delta_{n-j,0} I_q)_{j=0}^{n},$$

where $\delta_{j,k}$ is the Kronecker delta: $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$. Obviously, $T_{q,n}^* = [\delta_{j+1,k} I_q]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$.

It seems to be useful to recall well-known Lyapunov identities for block Hankel matrices. (These equations can be also easily proved by straightforward calculation.)

Remark 4.1. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices.*

(a) *For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then $H_n T_{q,n}^* - T_{p,n} H_n = u_n v_{q,n}^* - v_{p,n} w_n$ and $H_n T_{q,n} - T_{p,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{p,n} \mathfrak{w}_n$. In particular, if $p = q$ and if $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$, then $H_n T_{q,n}^* - T_{q,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n^*$ and $H_n T_{q,n} - T_{q,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{q,n} \mathfrak{u}_n^*$ for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.*

(b) *For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we have $H_{\alpha\triangleright n} = -\alpha H_n + K_n$, $v_{p,n} v_{p,n}^* H_n = \left[R_{T_{p,n}}(\alpha)\right]^{-1} H_n - T_{p,n} H_{\alpha\triangleright n}$, and, in the case that $p = q$ and $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$ hold true, moreover $H_{\alpha\triangleright n} T_{q,n}^* - T_{q,n} H_{\alpha\triangleright n} = (-\alpha u_n - y_{0,n}) v_{q,n}^* - v_{q,n}(-\alpha u_n - y_{0,n})^*$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.*

Remark 4.2. *For each $n \in \mathbb{N}_0$, the matrix-valued functions $R_{T_{q,n}} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ and $R_{T_{q,n}^*} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by $R_{T_{q,n}}(z) := (I_{(n+1)q} - zT_{q,n})^{-1}$ and $R_{T_{q,n}^*}(z) := (I_{(n+1)q} - zT_{q,n}^*)^{-1}$ are well-defined matrix polynomials of degree n, which can be represented, for each $z \in \mathbb{C}$, via $R_{T_{q,n}}(z) = \sum_{j=0}^{n} z^j T_{q,n}^j$ and $R_{T_{q,n}^*}(z) = \sum_{j=0}^{n} z^j (T_{q,n}^*)^j$, respectively. In particular, $R_{T_{q,n}^*}(z) = [R_{T_{q,n}}(\overline{z})]^*$ for all $z \in \mathbb{C}$.*

For each $n \in \mathbb{N}_0$, let $E_{q,n} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ and $F_{q,n} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ be defined by

$$E_{q,n}(z) := \operatorname{col}(z^j I_q)_{j=0}^{n} \qquad \text{and} \qquad F_{q,n}(z) := zE_{q,n}(z), \tag{8}$$

respectively. Obviously, for each $n \in \mathbb{N}_0$ and each $z \in \mathbb{C}$, we have $R_{T_{q,n}}(z) v_{q,n} = E_{q,n}(z)$.

Notation 4.3. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Further, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f \colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $P_{2n}^{[f]} \colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ be defined by*

$$P_{2n}^{[f]}(z) := \begin{bmatrix} H_n & R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n] \\ (R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n])^* & \frac{f(z) - f^*(z)}{z - \overline{z}} \end{bmatrix}. \tag{9}$$

If $\kappa \geq 1$, then, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $P^{[f]}_{2n+1}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ be given by

$$P^{[f]}_{2n+1}(z) := \begin{bmatrix} H_{\alpha\triangleright n} & R_{T_{q,n}}(z)(v_{q,n}[(z-\alpha)f(z)] -(-\alpha u_n - y_{0,n})) \\ [R_{T_{q,n}}(z)(v_{q,n}[(z-\alpha)f(z)] -(-\alpha u_n - y_{0,n}))]^* & \frac{(z-\alpha)f(z)-[(z-\alpha)f(z)]^*}{z-\overline{z}} \end{bmatrix}. \tag{10}$$

Furthermore, let $P^{[f]}_{-1}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by

$$P^{[f]}_{-1}(z) := \frac{(z-\alpha)f(z)-[(z-\alpha)f(z)]^*}{z-\overline{z}}.$$

With respect to the Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ if, $\mathcal{G} = \mathbb{C}$, then the functions (9) and (10) are called the Potapov fundamental matrix-valued functions connected to the Stieltjes moment problem (generated by f). If these matrices are both non-negative Hermitian, then one says that the Potapov's fundamental matrix inequalities for the function f are fulfilled.

Remark 4.4. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q\times q$ matrices, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G}\setminus\mathbb{R} \neq \emptyset$, and let $S\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Straightforward calculations show then that the following statements hold true:*

(a) *For every choice of $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $z \in \mathcal{G}\setminus\mathbb{R}$, we have*

$$\begin{bmatrix} s_0 & S(z) \\ S^*(z) & \frac{S(z)-S^*(z)}{z-\overline{z}} \end{bmatrix} = [v_{q,n+1}, \mathfrak{v}_{q,n+1}]^* P^{[S]}_{2n}(z)[v_{q,n+1}, \mathfrak{v}_{q,n+1}]. \tag{11}$$

(b) *If $\kappa \geq 1$, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and each $z \in \mathcal{G}\setminus\mathbb{R}$, then*

$$\begin{split}\begin{bmatrix} -\alpha s_0 + s_1 & (z-\alpha)S(z)+s_0 \\ [(z-\alpha)S(z)+s_0]^* & \frac{(z-\alpha)S(z)-[(z-\alpha)S(z)]^*}{z-\overline{z}} \end{bmatrix} = \\ = [v_{q,n+1}, \mathfrak{v}_{q,n+1}]^* P^{[S]}_{2n+1}(z)[v_{q,n+1}, \mathfrak{v}_{q,n+1}].\end{split} \tag{12}$$

Notation 4.5. *For each $n \in \mathbb{N}_0$, let $\tilde{A}_{2n}(z) := \operatorname{diag}([R_{T_{q,n}}(\overline{z})]^{-1}, I_q)$, let*

$$\tilde{B}_{2n}(z) := \begin{bmatrix} I_{(n+1)q} & (z-\overline{z})v_{q,n} \\ 0_{q\times(n+1)q} & I_q \end{bmatrix},$$

let $\tilde{C}_{2n}(z) := \operatorname{diag}(R_{T_{q,n}}(z), I_q)$, let

$$\tilde{A}_{2n+1}(z) := \tilde{A}_{2n}(z),$$

let $\tilde{B}_{2n+1}(z) := \tilde{B}_{2n}(z)$, and let $\tilde{C}_{2n+1}(z) := \tilde{C}_{2n}(z)$.

Lemma 4.6. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of Hermitian complex $q\times q$ matrices. Let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G}\setminus\mathbb{R}\neq\emptyset$. Further, let $f\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function, let $\mathcal{G}^\vee := \{z \in \mathbb{C}\colon \overline{z} \in \mathcal{G}\}$, and let $f^\vee\colon \mathcal{G}^\vee \to \mathbb{C}^{q\times q}$ be defined by $f^\vee(z) := f^*(\overline{z})$. For each $k \in \mathbb{Z}_{-1,\kappa}$ and each $z \in \mathcal{G}^\vee\setminus\mathbb{R}$, then $P^{[f^\vee]}_k(z) = X_k(z)P^{[f]}_k(\overline{z})X^*_k(z)$, where $X_k(z) := \tilde{C}_k(z)\tilde{B}_k(z)\tilde{A}_k(z)$.*

Taking into account Remark 4.1, Lemma 4.6 can be proved by straightforward calculations (, for details, see e. g. [30, Lemma 4.8]).

In the following, we will write $\mathfrak{B}_{p\times q}$ for the σ-algebra of all Borel subsets of $\mathbb{C}^{p\times q}$. Let $(\Omega,\mathfrak{A})$ be a measurable space and let $\mu\in\mathcal{M}_{\geq}^{q}(\Omega,\mathfrak{A})$. Then μ is absolutely continuous with respect to its trace measure $\tau:=\operatorname{tr}\mu$. Let μ_{τ}' be a version of the Radon–Nikodym derivative of μ with respect to τ. A pair $[\Phi,\Psi]$ of an $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable mapping $\Phi\colon\Omega\to\mathbb{C}^{p\times q}$ and an $\mathfrak{A}$-$\mathfrak{B}_{r\times q}$-measurable mapping $\Psi\colon\Omega\to\mathbb{C}^{r\times q}$ is called left-integrable with respect to μ if $\Phi\mu_{\tau}'\Psi^{*}$ belongs to $[\mathcal{L}^{1}(\Omega,\mathfrak{A},\tau;\mathbb{C})]^{p\times r}$. In this case, the corresponding integral is defined by $\int_{\Omega}\Phi\mathrm{d}\mu\Psi^{*}:=\int_{\Omega}\Phi\mu_{\tau}'\Psi^{*}\mathrm{d}\tau$ and we also use the notation $\int_{\Omega}\Phi(\omega)\mu(\mathrm{d}\omega)\Psi^{*}(\omega)$ for it. In the following, when we write such an integral $\int_{\Omega}\Phi\mathrm{d}\mu\Psi^{*}$, then we also mean that the pair $[\Phi,\Psi]$ is left-integrable with respect to μ. By $p\times q$-$\mathcal{L}^{2}(\Omega,\mathfrak{A},\mu;\mathbb{C})$ we denote the set of all $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable mappings for which the pair $[\Phi,\Phi]$ is left-integrable which respect to μ. Furthermore, for each subset A of Ω, we will use 1_{A} to denote the indicator function of the set A (defined on Ω).

Remark 4.7. *Let $\Omega\in\mathfrak{B}_{\mathbb{R}}\setminus\{\emptyset\}$, let $m\in\mathbb{N}_{0}$, and let $\sigma\in\mathcal{M}_{\geq}^{q}(\Omega)$. In view of Lemma 7.2, it is readily checked that σ belongs to $\mathcal{M}_{\geq,2m}^{q}(\Omega)$ if and only if $\operatorname{Rstr}_{\Omega}E_{q,m}$ belongs to $(m+1)q\times q$-$\mathcal{L}^{2}(\Omega,\mathfrak{B}_{\Omega},\sigma;\mathbb{C})$, where $E_{q,m}$ is given by (8). If $\sigma\in\mathcal{M}_{\geq,2m}^{q}(\Omega)$, then Lemma 7.2 also shows that, for each $n\in\mathbb{N}_{0}$ with $n\leq m$, the block Hankel matrix $H_{n}^{[\sigma]}:=[s_{j+k}^{[\sigma]}]_{j,k=0}^{n}$ admits the integral representation*

$$H_{n}^{[\sigma]}=\int_{\Omega}E_{q,n}(t)\sigma(\mathrm{d}t)E_{q,n}^{*}(t).\tag{13}$$

If $\alpha\in\mathbb{R}$, if $\kappa\in\mathbb{N}\cup\{\infty\}$, and if $\sigma\in\mathcal{M}_{\geq,\kappa}^{q}([\alpha,\infty))$, then let $H_{\alpha\triangleright n}^{[\sigma]}:=[s_{\alpha\triangleright j+k}^{[\sigma]}]_{j,k=0}^{n}$ for each $n\in\mathbb{N}_{0}$ with $2n+1\leq\kappa$.

Remark 4.8. *Let $\alpha\in\mathbb{R}$ and let $\sigma\in\mathcal{M}_{\geq,1}^{q}([\alpha,\infty))$. Using Proposition 7.4 and Remark 7.3, it is readily checked that the following statements hold true:*

(a) *The function $\phi\colon[\alpha,\infty)\to\mathbb{C}^{q\times q}$ defined by $\phi(t):=\sqrt{t-\alpha}I_{q}$ belongs to $q\times q$-$\mathcal{L}^{2}([\alpha,\infty),\mathfrak{B}_{[\alpha,\infty)},\sigma;\mathbb{C})$ and $\sigma^{\#}\colon\mathfrak{B}_{[\alpha,\infty)}\to\mathbb{C}^{q\times q}$ given by*

$$\sigma^{\#}(B):=\int_{B}(\sqrt{t-\alpha}I_{q})\sigma(\mathrm{d}t)(\sqrt{t-\alpha}I_{q})^{*}\tag{14}$$

belongs to $\mathcal{M}_{\geq}^{q}([\alpha,\infty))$.

(b) *If $n\in\mathbb{N}_{0}$ and if $\sigma\in\mathcal{M}_{\geq,2n+1}^{q}([\alpha,\infty))$, then*

$$H_{\alpha\triangleright n}^{[\sigma]}=\int_{[\alpha,\infty)}\big[\sqrt{t-\alpha}E_{q,n}(t)\big]\sigma(\mathrm{d}t)\big[\sqrt{t-\alpha}E_{q,n}(t)\big]^{*}.\tag{15}$$

(c) *If $n\in\mathbb{N}_{0}$ and if $\sigma^{\#}\in\mathcal{M}_{\geq,2n}^{q}([\alpha,\infty))$, then σ belongs to $\mathcal{M}_{\geq,2n+1}^{q}([\alpha,\infty))$ and furthermore $s_{j}^{[\sigma^{\#}]}=s_{j+1}^{[\sigma]}-\alpha s_{j}^{[\sigma]}$ for all $j\in\mathbb{Z}_{0,2n}$ and $H_{n}^{[\sigma^{\#}]}=H_{\alpha\triangleright n}^{[\sigma]}$.*

The next proposition shows that each solution of problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ fulfills necessarily the system of the corresponding Potapov's fundamental matrix inequalities.

Proposition 4.9. *Let $\alpha\in\mathbb{R}$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q\times q$ matrices such that $\mathcal{M}_{\geq}^q[[\alpha,\infty);(s_j)_{j=0}^m,\leq]\neq\emptyset$. Let $\sigma\in\mathcal{M}_{\geq}^q[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ and let S be the $[\alpha,\infty)$-Stieltjes transform of σ. For each $j\in\mathbb{Z}_{0,m}$, let $s_j^{[\sigma]}$ be given by* (1)*. Then*

$$P_{2n}^{[S]}(z)=\int_{[\alpha,\infty)}\begin{bmatrix}E_{q,n}(t)\\ \frac{1}{t-z}I_q\end{bmatrix}\sigma(\mathrm{d}t)\begin{bmatrix}E_{q,n}(t)\\ \frac{1}{t-z}I_q\end{bmatrix}^*+\begin{bmatrix}\mathfrak{v}_{q,n}\\ 0_{q\times q}\end{bmatrix}(s_{2n}-s_{2n}^{[\sigma]})\begin{bmatrix}\mathfrak{v}_{q,n}\\ 0_{q\times q}\end{bmatrix}^*$$

for each $n\in\mathbb{N}_0$ with $2n\leq m$ and all $z\in\mathbb{C}\setminus\mathbb{R}$, where $E_{q,n}$ is given by (8)*, and*

$$\begin{aligned}&P_{2n+1}^{[S]}(z)=\\ &\quad=\int_{[\alpha,\infty)}\left(\sqrt{t-\alpha}\begin{bmatrix}E_{q,n}(t)\\ \frac{1}{t-z}I_q\end{bmatrix}\right)\sigma(\mathrm{d}t)\left(\sqrt{t-\alpha}\begin{bmatrix}E_{q,n}(t)\\ \frac{1}{t-z}I_q\end{bmatrix}\right)^*+\\ &\quad+\begin{bmatrix}\mathfrak{v}_{q,n}\\ 0_{q\times q}\end{bmatrix}(s_{2n+1}-s_{2n+1}^{[\sigma]})\begin{bmatrix}\mathfrak{v}_{q,n}\\ 0_{q\times q}\end{bmatrix}^*\end{aligned}$$

for each $n\in\mathbb{N}_0$ with $2n+1\leq m$ and all $z\in\mathbb{C}\setminus\mathbb{R}$. In particular, for every choice of $k\in\mathbb{Z}_{0,m}$ and $z\in\mathbb{C}\setminus\mathbb{R}$, the matrix $P_k^{[S]}(z)$ is non-negative Hermitian.

Proposition 4.9 can be proved using standard arguments of integration theory of non-negative Hermitian measures (Lemma 7.2 and Remark 7.3). We omit the details.

5. Some integral estimates for the scalar case

In this section, we state some integral representations and estimates in the scalar case $q=1$.

Lemma 5.1. *Let $\alpha\in\mathbb{R}$ and let $F\in\mathcal{R}_1(\Pi_+)$ with Nevanlinna parametrization (A,B,ν) and spectral measure μ. Then:*

(a) *For each $w\in\Pi_+$, the integral $\int_{\mathbb{R}}|t-w|^{-2}\mu(\mathrm{d}t)$ is finite and*

$$\Im F(w)=(\Im w)\left[B+\int_{\mathbb{R}}\frac{1}{|t-w|^2}\mu(\mathrm{d}t)\right].\tag{16}$$

(b) *For each $w\in\Pi_+$, the integral $\int_{\mathbb{R}}|t[|t-w|^{-2}-(1+t^2)^{-1}]-\alpha|t-w|^{-2}|\mu(\mathrm{d}t)$ is finite and $F^{\#}\colon\Pi_+\to\mathbb{C}$ defined by*

$$F^{\#}(w):=(w-\alpha)F(w)\tag{17}$$

satisfies, for each $w \in \Pi_+$, the equation

$$\begin{aligned}\Im F^{\#}(w) &= \\ &= (\Im w)\Bigg(A + B(2\Re w - \alpha) + \\ &\quad + \int_{\mathbb{R}} \left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right] \mu(\mathrm{d}t) \Bigg).\end{aligned} \tag{18}$$

Proof. In view of

$$\int_{\mathbb{R}} \frac{1}{1+t^2} \mu(\mathrm{d}t) = \int_{\mathbb{R}} \frac{1}{1+t^2}(1+t^2)\nu(\mathrm{d}t) = \nu(\mathbb{R}) < \infty,$$

we see that, for each $w \in \Pi_+$, the function $\psi_w \colon \mathbb{R} \to \mathbb{C}$ given by the equation $\psi_w(t) := (t-w)^{-1} - t(1+t^2)^{-1}$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu; \mathbb{C})$. By virtue of a result due to R. Nevanlinna (see, e. g. [47, Theorem A.2]), for each $w \in \Pi_+$, we have

$$F(w) = A + Bw + \int_{\mathbb{R}} \left(\frac{1}{t-w} - \frac{t}{1+t^2} \right) \mu(\mathrm{d}t). \tag{19}$$

(a) Let $w \in \Pi_+$. For each $t \in \mathbb{R}$, then $\Im \psi_w(t) = (\Im w)|t-w|^{-2}$. Thus,

$$\int_{\mathbb{R}} \left| \frac{1}{|t-w|^2} \right| \mu(\mathrm{d}t) = \frac{1}{\Im w} \int_{\mathbb{R}} \Im \psi_w(t) \mu(\mathrm{d}t) \le \frac{1}{\Im w} \int_{\mathbb{R}} |\psi_w(t)| \mu(\mathrm{d}t) < \infty$$

and

$$\Im \left[\int_{\mathbb{R}} \psi_w(t) \mu(\mathrm{d}t) \right] = \int_{\mathbb{R}} \Im \psi_w(t) \mu(\mathrm{d}t) = (\Im w) \int_{\mathbb{R}} \frac{1}{|t-w|^2} \mu(\mathrm{d}t). \tag{20}$$

Because of $A \in \mathbb{R}$ and $B \in [0, \infty)$, we have $\Im A = 0$ and $\Im(wB) = (\Im w)B$. Consequently, from (19), and (20) we get then (16).

(b) Let $w \in \Pi_+$. In view of (17) and (19), we obtain

$$F^{\#}(w) = A(w-\alpha) + Bw(w-\alpha) + \int_{\mathbb{R}} \left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2} \right] \mu(\mathrm{d}t). \tag{21}$$

For each $t \in \mathbb{R}$, we see that $(w-\alpha)\psi_w(t) = (w-\alpha)/(t-w) - t(w-\alpha)/(1+t^2)$ holds true. Hence, $(w-\alpha)\psi_w \in \mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu; \mathbb{C})$ and, for each $t \in \mathbb{R}$, we have furthermore $\Im[(w-\alpha)\psi_w(t)] = 2\mathrm{i}(\Im w)[t(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}) - \frac{\alpha}{|t-w|^2}]$. This implies

$$\int_{\mathbb{R}} \left| t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right| \mu(\mathrm{d}t) \le \frac{1}{\Im w} \int_{\mathbb{R}} |(w-\alpha)\psi_w(t)| \mu(\mathrm{d}t) < \infty$$

and

$$\begin{aligned}\Im \left[\int_{\mathbb{R}} (w-\alpha)\psi_w(t) \mu(\mathrm{d}t) \right] &= \\ = (\Im w) \int_{\mathbb{R}} \left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2} \right) - \frac{\alpha}{|t-w|^2} \right] \mu(\mathrm{d}t).\end{aligned} \tag{22}$$

Obviously, $\Im(w^2) = 2(\Re w)(\Im w)$. Hence, $\Im[w(w-\alpha)] = \Im(w^2) - \Im(w\alpha) = (\Im w)(2\Re w - \alpha)$. Thus, $\Im[Bw(w-\alpha)] = B(\Im w)(2\Re w - \alpha)$. Then, by virtue of (21), and (22), we get (18) from

$$\begin{aligned}\Im F^{\#}(w) &= \Im\left(A(w-\alpha) + Bw(w-\alpha) + \int_{\mathbb{R}}\left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2}\right]\mu(\mathrm{d}t)\right)\\ &= \Im[A(w-\alpha)] + \Im[Bw(w-\alpha)] + \Im\left(\int_{\mathbb{R}}\left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2}\right]\mu(\mathrm{d}t)\right)\\ &= A\Im w + B(\Im w)(2\Re w - \alpha) +\\ &+ (\Im w)\int_{\mathbb{R}}\left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}\right) - \frac{\alpha}{|t-w|^2}\right]\mu(\mathrm{d}t).\end{aligned}$$

□

Remark 5.2. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ with spectral measure μ. Further, let $\ell_1, \ell_2 \in \mathbb{R}$ be such that $\ell_1 < \ell_2 < \alpha$. Then it is readily checked that for every choice of $a \in (-\infty, \ell_1)$ and $b \in (\ell_2, \infty)$, there exists a $K_{a,b} \in \mathbb{R}$ such that, for each $x \in [\ell_1, \ell_2]$, the inequality $\int_{\mathbb{R}\setminus(a,b)}(t-x)^{-2}\mu(\mathrm{d}t) < K_{a,b}$ holds true.*

Remark 5.3. *Let $r, s \in \mathbb{R}$. Then it is readily checked that the following statements hold true:*

(a) *If $r < s$ and $s \neq 0$, then there exists a number $a \in (-\infty, r) \cap (-\infty, 0)$ such that*

$$\left|t\left[\frac{1}{(t-x)^2+y^2} - \frac{1}{1+t^2}\right]\right| < \left(2 + \left|\frac{r}{s}\right|\right) \cdot \left|t\left[\frac{1}{(t-s)^2+1} - \frac{1}{1+t^2}\right]\right| \tag{23}$$

is valid for every choice of $x \in [r, s]$ and $y \in (0,1)$ and $t \in (-\infty, a]$.

(b) *If $s < r$ and $r \neq 0$, then there exists a number $b \in (r, \infty) \cap (0, \infty)$ such that, for every choice of $x \in [s, r]$ and $y \in (0,1)$ and $t \in [b, \infty)$, inequality (23) holds true.*

Lemma 5.4. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ with spectral measure μ. Further, let ℓ_1 and ℓ_2 be real numbers with $\ell_1 < \ell_2 < \alpha$. Then there are real numbers a, b, and C with $a < \ell_1$ and $\ell_2 < b < \alpha$ such that $\int_{\mathbb{R}\setminus(a,b)}|t[\frac{1}{(t-x)^2+y^2} - \frac{1}{1+t^2}] - \frac{\alpha}{(t-x)^2+y^2}|\mu(\mathrm{d}t) < C$ holds true for every choice of $x \in [\ell_1, \ell_2]$ and $y \in (0,1)$.*

Using Lemma 5.1 and Remarks 5.2 and 5.3, Lemma 5.4 can be proved analogous to the well-known special case $\alpha = 0$. However, in the general case of on arbitrary real number α, these straightforward calculations are very lengthy. We omit the details.

Lemma 5.5. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^{\#}\colon \Pi_+ \to \mathbb{C}$ defined by (17) belongs to $\mathcal{R}_1(\Pi_+)$. Further, let μ be the spectral measure of F and let ℓ_1 and ℓ_2 be real numbers with $\ell_1 < \ell_2 < \alpha$. Then there are real numbers a, b, and C with $a < \ell_1$ and $\ell_2 < b < \alpha$ such that $\int_{(a,b)}|\frac{t-\alpha}{(t-x)^2+y^2}|\sigma(\mathrm{d}t) < C$ and $\int_{(a,b)}|\frac{1}{(t-x)^2}|\sigma(\mathrm{d}t) < C$ hold true for every choice of $x \in [\ell_1, \ell_2]$ and $y \in [0, \infty)$.*

Lemma 5.5 can be proved, using Lemmata 5.1 and 5.4 and Beppo Levi's Theorem of monotone convergence. We omit the details.

Remark 5.6. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^{\#}: \Pi_+ \to \mathbb{C}$ defined by* (17) *belongs to $\mathcal{R}_1(\Pi_+)$. Let μ be the spectral measure of F and let ℓ_1 and ℓ_2 be real numbers with $\ell_1 < \ell_2 < \alpha$. Then one can easily see from Remark 5.2 and Lemma 5.5 that there is a real number C such that $\int_{\mathbb{R}} (t-x)^{-2} \mu(\mathrm{d}t) < C$ for all $x \in [\ell_1, \ell_2]$.*

Lemma 5.7. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^{\#}: \Pi_+ \to \mathbb{C}$ defined by* (17) *belongs to $\mathcal{R}_1(\Pi_+)$. Then the Nevanlinna measure ν of F and the spectral measure μ of F fulfill $\nu((-\infty, \alpha)) = 0$ and $\mu((-\infty, \alpha)) = 0$.*

Proof. (I) In the first step of the proof, we consider arbitrary real numbers ℓ_1 and ℓ_2 with $\ell_1 < \ell_2 < \alpha$. Let (A, B, ν) be the Nevanlinna parametrization of F. Because of Remark 5.6, there is a $C \in \mathbb{R}$ such that $\int_{\mathbb{R}} (t-x)^{-2} \mu(\mathrm{d}t) < C$ is true for all $x \in [\ell_1, \ell_2]$. Since F belongs to $\mathcal{R}_1(\Pi_+)$, for each $x \in [\ell_1, \ell_2]$ and each $\epsilon \in (0, \infty)$, from Lemma 5.1 we get then $0 \le \Im F(x + \mathrm{i}\epsilon) = \epsilon(B + \int_{\mathbb{R}} [(t-x)^2 + \epsilon^2]^{-1} \mu(\mathrm{d}t)) < \epsilon(B + C)$ and, consequently,

$$0 \le \int_{[\ell_1, \ell_2]} \Im F(x + \mathrm{i}\epsilon) \lambda^{(1)}(\mathrm{d}x) \le \epsilon(B + C)(\ell_2 - \ell_1), \tag{24}$$

where $\lambda^{(1)}$ is the Lebesgue measure defined on $\mathfrak{B}_{\mathbb{R}}$. In view of $F \in \mathcal{R}_1(\Pi_+)$, the inversion formula of Stieltjes–Perron (see, e. g. [47, Appendix, p. 390]) yields

$$\frac{1}{2}[\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma((\ell_1, \ell_2)) = \frac{1}{\pi} \lim_{\epsilon \to 0+0} \int_{[\ell_1, \ell_2]} \Im F(x + \mathrm{i}\epsilon) \lambda^{(1)}(\mathrm{d}x). \tag{25}$$

Combining (25) and (24), we obtain $\sigma((\ell_1, \ell_2)) = 0$, from

$$\begin{aligned} 0 \le \sigma((\ell_1, \ell_2)) &\le \frac{1}{2}[\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma((\ell_1, \ell_2)) \\ &= \frac{1}{\pi} \lim_{\epsilon \to 0+0} \int_{[\ell_1, \ell_2]} \Im F(x + \mathrm{i}\epsilon) \lambda^{(1)}(\mathrm{d}x) \le \frac{1}{\pi} \lim_{\epsilon \to 0+0} [\epsilon(B + C)(\ell_2 - \ell_1)] = 0. \end{aligned}$$

(II) For each $n \in \mathbb{N}$, the real numbers $a_n := \alpha - (1 + n)$ and $b_n := \alpha - \frac{1}{n}$ fulfill $a_n < b_n < \alpha$. Thus, part (I) of the proof provides us $\mu((a_n, b_n)) = 0$. Obviously, $(a_n, b_n) \subseteq (a_{n+1}, b_{n+1})$ for each $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} (a_n, b_n) = (-\infty, \alpha)$. Hence, $\mu((-\infty, \alpha)) = \lim_{n \to \infty} \mu((a_n, b_n)) = 0$. Thus, $\nu((-\infty, \alpha)) = 0$ follows from

$$\begin{aligned} 0 \le \nu((-\infty, \alpha)) = \int_{(-\infty, \alpha)} 1 \nu(\mathrm{d}t) \le \\ \le \int_{(-\infty, \alpha)} (1 + t^2) \nu(\mathrm{d}t) = \mu((-\infty, \alpha)) = 0. \end{aligned}$$

□

6. From the system of Potapov's fundamental matrix inequalities to the moment problem

Proposition 4.9 showed that the Stieltjes transform of an arbitrary solution of problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^m, \le]$ fulfills necessarily the system of corresponding Potapov's fundamental matrix inequalities. In this section, we are going to prove that the validity of the system of Potapov's fundamental matrix inequalities for a holomorphic $q \times q$ matrix-valued function defined on $\mathbb{C} \setminus [\alpha, \infty)$ is also sufficient to be the Stieltjes transform of some solution of this matricial Stieltjes-type moment problem. For the convenience of the reader, first we state two well-known facts.

Remark 6.1. *Let $\mathcal{D}$ be a discrete subset of Π_+ and let $F\colon \Pi_+ \setminus \mathcal{D} \to \mathbb{C}^{q\times q}$ be a matrix-valued function which is holomorphic in $\Pi_+ \setminus \mathcal{D}$ and which fulfills $\Im F(z) \in \mathbb{C}^{q\times q}_{\geq}$ for all $z \in \Pi_+ \setminus \mathcal{D}$. Then one can easily see from [16, Lemma 2.1.9] that there is a function $F^{\triangle} \in \mathcal{R}_q(\Pi_+)$ such that $\operatorname{Rstr}_{\Pi_+ \setminus \mathcal{D}} F^{\triangle} = F$.*

Remark 6.2. *Let $A, B \in \mathbb{C}^{q\times q}$, let M be an open subset of $\mathbb{R}$, and let $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R} \setminus M)$. In view of a well-known result on integrals which depend on a complex parameter (see, e. g. [24, Satz 5.8]), it is readily checked that $\phi\colon \Pi_+ \cup M \cup \Pi_- \to \mathbb{C}^{q\times q}$ given by*

$$\phi(z) := A + Bz + \int_{\mathbb{R}\setminus M} \frac{1+tz}{t-z} \nu(\mathrm{d}t)$$

is holomorphic in $\Pi_+ \cup M \cup \Pi_-$.

In the following, for all $\alpha \in \mathbb{R}$, let $\mathbb{C}_{\alpha,-} := \{z \in \mathbb{C}\colon \Re z \in (-\infty, \alpha)\}$.

Lemma 6.3. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_q(\Pi_+)$ be such that $F^{\#}\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by $F^{\#}(w) := (w-\alpha)F(w)$ belongs to $\mathcal{R}_q(\Pi_+)$. Further, let ν be the Nevanlinna measure of F. Then $\nu((-\infty, \alpha)) = 0$ and the following two statements hold true:*

(a) *There is a function $F_\alpha\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ such that $\operatorname{Rstr}_{\Pi_+} F_\alpha = F$ and $F_\alpha((-\infty, \alpha)) \subseteq \mathbb{C}^{q\times q}_{\mathrm{H}}$ are fulfilled.*
(b) *There exists a unique function $S \in \mathcal{S}_{q;[\alpha,\infty)}$ with $\operatorname{Rstr}_{\Pi_+} S = F$.*

Proof. Since F and $F^{\#}$ belong to $\mathcal{R}_q(\Pi_+)$, for all $u \in \mathbb{C}^q$, we see that $\{u^* F u, u^* F^{\#} u\} \subseteq \mathcal{R}_1(\Pi_+)$ and that $u^*\nu u$ is the Nevanlinna measure of $u^* F u$. Because of Lemma 5.7, for all $u \in \mathbb{C}^q$, we have $u^*\nu((-\infty, \alpha))u = (u^*\nu u)((-\infty, \alpha)) = 0 = u^* 0_{q\times q} u$. Hence, $\nu((-\infty, \alpha)) = 0_{q\times q}$.

(a) Obviously, $\tilde{\nu} := \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \nu$ belongs to $\mathcal{M}^q_{\geq}([\alpha, \infty))$. By virtue of $F \in \mathcal{R}_q(\Pi_+)$ and Theorem 3.1, there are matrices $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $B \in \mathbb{C}^{q\times q}_{\geq}$ such that (3) holds true for each $z \in \Pi_+$. Remark 6.2 shows that $F_\alpha\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ given by

$$F_\alpha(z) := A + Bz + \int_{[\alpha,\infty)} \frac{1+tz}{t-z} \tilde{\nu}(\mathrm{d}t) \tag{26}$$

is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Comparing (3) and (26), we get $F_\alpha(z) = F(z)$ for each $z \in \Pi_+$. For every choice of $x \in \mathbb{R}$, we have

$$\left[\int_{[\alpha,\infty)} \frac{1+tx}{t-x} \tilde{\nu}(\mathrm{d}t)\right]^* = \int_{[\alpha,\infty)} \overline{\left(\frac{1+tx}{t-x}\right)} \tilde{\nu}(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{1+tx}{t-x} \tilde{\nu}(\mathrm{d}t).$$

In view of (26), $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, and $B \in \mathbb{C}_{\geq}^{q\times q}$, then $[F_\alpha(x)]^* = F_\alpha(x)$ follows for each $x \in (-\infty, \alpha)$.

(b) Because of part (a), there is a holomorphic function $S\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ such that

$$\operatorname{Rstr}_{\Pi_+} S = F \qquad \text{and} \qquad S((-\infty,\alpha)) \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q} \tag{27}$$

hold true. According to $\{F, F^\#\} \subseteq \mathcal{R}_q(\Pi_+)$ and (27), for all $z \in \Pi_+$, then

$$\Im S(z) = \Im F(z) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{and} \quad \Im[(z-\alpha)S(z)] = \Im F^\#(z) \in \mathbb{C}_{\geq}^{q\times q}. \tag{28}$$

For all $z \in \mathbb{C}_{\alpha,-} \cap \Pi_+$, we have $\Im[(z-\alpha)S(z)] = [\Re(z-\alpha)]\Im S(z) + (\Im z)\Re S(z)$ and, by virtue of (28), consequently,

$$\Re S(z) = \frac{\Im[(z-\alpha)S(z)]}{\Im z} + [-\Re(z-\alpha)]\frac{\Im S(z)}{\Im z} \in \mathbb{C}_{\geq}^{q\times q}. \tag{29}$$

Now we consider an arbitrary monotonically nondecreasing sequence $(y_n)_{n=1}^\infty$ of positive real numbers with $\lim_{n\to\infty} y_n = 0$. Since the function S is holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$, the functions $\Re S$ and $\Im S$ are continuous in $\mathbb{C}\setminus[\alpha,\infty)$. Thus, for each $x \in (-\infty,\alpha)$, we have $x + \mathrm{i}y_n \in \mathbb{C}_{\alpha,-} \cap \Pi_+$ for all $n \in \mathbb{N}$ and, hence, because of (29), and (28), then

$$\begin{aligned} \Re S(x) &= \lim_{n\to\infty} \Re S(x+\mathrm{i}y_n) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{and} \\ \Im S(x) &= \lim_{n\to\infty} \Im S(x+\mathrm{i}y_n) \in \mathbb{C}_{\geq}^{q\times q}. \end{aligned} \tag{30}$$

Combining (27) and (30), for each $x \in (-\infty,\alpha)$, we get $\Re S(x) + \mathrm{i}\Im S(x) = S(x) = [S(x)]^* = \Re S(x) - \mathrm{i}\Im S(x)$ and, hence, $\Im S(x) = 0$. From (30) then $S(x) \in \mathbb{C}_{\geq}^{q\times q}$ follows for each $x \in (-\infty,\alpha)$. Consequently, $S \in \mathcal{S}_{q;[\alpha,\infty)}$. Now we consider an arbitrary $S^\square \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+} S^\square = F$. From (27) we get then $S^\square(z) = F(z) = S(z)$ for each $z \in \Pi_+$. Thus, the identity theorem for holomorphic functions provides us $S^\square = S$. □

Proposition 6.4. *Let $\alpha \in \mathbb{R}$ and let $\mathcal{D}$ be a discrete subset of Π_+. Let $F\colon \Pi_+ \setminus \mathcal{D} \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function and let $F^\#\colon \Pi_+ \to \mathbb{C}^{q\times q}$ be defined by $F^\#(w) := (w-\alpha)F(w)$. Suppose $\{\Im F(w), \Im F^\#(w)\} \subseteq \mathbb{C}_{\geq}^{q\times q}$ for all $w \in \Pi_+ \setminus \mathcal{D}$. Then there is a unique $S \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S = F$.*

Proposition 6.4 can be easily proved using Remark 6.1, Lemma 6.3, and the identity theorem for holomorphic functions. We omit the details.

Theorem 6.5. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q\times q$ matrices, and let $m \in \mathbb{Z}_{0,\kappa}$. Further, let $\mathcal{D}$ be a discrete subset of Π_+ and let $F\colon \Pi_+ \setminus \mathcal{D} \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function such that*

$$P_m^{[F]}(z) \geq 0 \qquad \textit{and} \qquad P_{m-1}^{[F]}(z) \geq 0 \qquad \textit{for each } z \in \Pi_+ \setminus \mathcal{D}. \tag{31}$$

Then there exists a unique $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S = F$. Moreover, the inequality $P_k^{[S]}(z) \geq 0$ holds true for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof. From (31) and Notation 4.3 we see that $H_n \geq 0$ for each $n \in \mathbb{N}_0$ with $2n \leq m$, that $H_{\alpha \triangleright n} \geq 0$ for each $n \in \mathbb{N}$ with $2n+1 \leq m$, that in particular $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,m}$, and that $\Im F(z) = (\Im z)\frac{F(z)-F^*(z)}{z-\overline{z}} \geq 0$ and $\Im[(z-\alpha)F(z)] = (\Im z)\frac{(z-\alpha)F(z)-[(z-\alpha)F(z)]^*}{z-\overline{z}} \geq 0$ hold true for each $z \in \Pi_+ \setminus \mathcal{D}$. Thus, because of Proposition 6.4, there exists a unique $S \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+ \setminus \mathcal{D}} S = F$. By continuity arguments, from (31) we get then $\{P_m^{[S]}(z), P_{m-1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{q \times q}$ for each $z \in \Pi_+$ and, consequently,

$$P_k^{[S]}(z) \geq 0 \qquad \text{for each } k \in \mathbb{Z}_{-1,m} \text{ and each } z \in \Pi_+. \tag{32}$$

In particular, $\tilde{S} := \operatorname{Rstr}_{\Pi_+} S$ fulfills

$$\begin{bmatrix} s_0 & \tilde{S}(z) \\ \tilde{S}^*(z) & \frac{\tilde{S}(z)-\tilde{S}^*(z)}{z-\overline{z}} \end{bmatrix} = P_0^{[S]}(z) \geq 0 \qquad \text{for each } z \in \Pi_+.$$

Consequently, Lemma 3.3 provides us $\tilde{S} \in \mathcal{R}'_{0,q}(\Pi_+)$ and $\sup_{y \in [1,\infty)} y \|S(\mathrm{i}y)\|_{\mathrm{S}} < \infty$. Hence, S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$. Then Theorem 3.5 shows that there is a $\sigma \in \mathcal{M}_{\geq}^q([\alpha,\infty))$ such that (7) holds true. Let $\tilde{S}^{\vee} \colon \Pi_- \to \mathbb{C}^{q \times q}$ be defined by $\tilde{S}^{\vee}(z) := S^*(\overline{z})$. Thus, from (7) we get $\tilde{S}^{\vee}(z) = [\int_{[\alpha,\infty)} \frac{1}{t-\overline{z}} \sigma(\mathrm{d}t)]^* = \int_{[\alpha,\infty)} \frac{1}{t-z} \sigma(\mathrm{d}t) = S(z)$ for each $z \in \Pi_-$. From (32) and Lemma 4.6 we see then that, for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$, there exists a matrix $X_k(z)$ such that $P_k^{[S]}(z) = P_k^{[\tilde{S}^{\vee}]}(z) = X_k(z) P_k^{[S]}(\overline{z}) X_k^*(z)$ is fulfilled for all $k \in \mathbb{Z}_{-1,m}$ and all $z \in \Pi_-$. In view of (32), this implies $P_k^{[S]}(z) \geq 0$ for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$. Because of $\mathbb{C} \setminus \mathbb{R} = \Pi_+ \cup \Pi_-$, the proof is complete. □

Remark 6.6. *For each $n \in \mathbb{N}_0$ and every choice of w and z in $\mathbb{C}$, it is readily checked that*

$$(z - \overline{w}) \left[R_{T_{q,n}^*}(w) \right]^* T_{q,n} R_{T_{q,n}}(z) = R_{T_{q,n}}(z) - \left[R_{T_{q,n}^*}(w) \right]^*.$$

Lemma 6.7. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. Then*

$$\begin{aligned} &H_n T_{q,n}^* R_{T_{q,n}^*}(z) - \left[R_{T_{q,n}^*}(w) \right]^* T_{q,n} H_n + \\ &+ \left[R_{T_{q,n}^*}(w) \right]^* (v_{q,n} u_n^* - u_n v_{q,n}^*) R_{T_{q,n}^*}(z) = \\ &= (z - \overline{w}) \left[R_{T_{q,n}^*}(w) \right]^* T_{q,n} H_n T_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{33}$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and every choice of w and z in $\mathbb{C}$. Furthermore,

$$\begin{aligned} &H_{\alpha \triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) - \left[R_{T_{q,n}^*}(w) \right]^* T_{q,n} H_{\alpha \triangleright n} + \\ &+ \left[R_{T_{q,n}^*}(w) \right]^* \left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v_{q,n}^* \right] R_{T_{q,n}^*}(z) = \\ &= (z - \overline{w}) \left[R_{T_{q,n}^*}(w) \right]^* T_{q,n} H_{\alpha \triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{34}$$

for all $\alpha \in \mathbb{R}$, all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, and every choice of w and z in $\mathbb{C}$.

Proof. By virtue of Remark 4.1(a), we have

$$
\begin{aligned}
&H_n T^*_{q,n} R_{T^*_{q,n}}(z) - \Big[R_{T^*_{q,n}}(w)\Big]^* T_{q,n} H_n + \\
&\quad + \Big[R_{T^*_{q,n}}(w)\Big]^* (v_{q,n} u_n^* - u_n v^*_{q,n}) R_{T^*_{q,n}}(z) = \\
&= \Big[R_{T^*_{q,n}}(w)\Big]^* \Bigg(\Big[R_{T^*_{q,n}}(w)\Big]^{-*} H_n T^*_{q,n} - T_{q,n} H_n R^{-1}_{T^*_{q,n}}(z) + \\
&\quad + (v_{q,n} u_n^* - u_n v^*_{q,n}) \Bigg) R_{T^*_{q,n}}(z) = \Big[R_{T^*_{q,n}}(w)\Big]^* \big[(I_{(n+1)q} - \overline{w} T_{q,n}) H_n T^*_{q,n} - \\
&\quad - T_{q,n} H_n (I_{(n+1)q} - z T^*_{q,n}) + (v_{q,n} u_n^* - u_n v^*_{q,n}) bigr] R_{T^*_{q,n}}(z) = \\
&= \Big[R_{T^*_{q,n}}(w)\Big]^* \Big[(z - \overline{w}) T_{q,n} H_n T^*_{q,n} - (T_{q,n} H_n - H_n T^*_{q,n}) + \\
&\quad + (v_{q,n} u_n^* - u_n v^*_{q,n})\Big] R_{T^*_{q,n}}(z) = \Big[R_{T^*_{q,n}}(w)\Big]^* \big[(z - \overline{w}) T_{q,n} H_n T^*_{q,n}\big] R_{T^*_{q,n}}(z) = \\
&= (z - \overline{w}) \Big[R_{T^*_{q,n}}(w)\Big]^* T_{q,n} H_n T^*_{q,n} R_{T^*_{q,n}}(z).
\end{aligned}
$$

Using Remark 4.1(b), equation (34) can be proved analogous to (33). □

Notation 6.8. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. For each $n \in \mathbb{N}_0$ with $2n \le \kappa$, let $F_{2n}\colon \mathcal{G} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by*

$$F_{2n}(z) := H_n T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) \tag{35}$$

and let $Q^{[f]}_{2n}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q\times(2n+2)q}$ be defined by

$$Q^{[f]}_{2n}(z) := \begin{bmatrix} H_n & F_{2n}(z) \\ F^*_{2n}(z) & \frac{F_{2n}(z) - F^*_{2n}(z)}{z - \overline{z}} \end{bmatrix}. \tag{36}$$

If $\kappa \ge 1$, then, for all $n \in \mathbb{N}_0$ with $2n+1 \le \kappa$, let $F_{2n+1}\colon \mathcal{G} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by

$$
\begin{aligned}
F_{2n+1}(z) := H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z)\big[v_{q,n}(z-\alpha) f(z) - \\
- (-\alpha u_n - y_{0,n})\big] v^*_{q,n} R_{T^*_{q,n}}(z)
\end{aligned} \tag{37}
$$

and let $Q^{[f]}_{2n+1}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q\times(2n+2)q}$ be defined by

$$Q^{[f]}_{2n+1}(z) := \begin{bmatrix} H_{\alpha\triangleright n} & F_{2n+1}(z) \\ F^*_{2n+1}(z) & \frac{F_{2n+1}(z) - F^*_{2n+1}(z)}{z - \overline{z}} \end{bmatrix}. \tag{38}$$

Further, for each $k \in \mathbb{N}_0$, let $m_{2k} := k$ and $m_{2k+1} := k$.

Proposition 6.9. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. Let $f\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Further, for each $k \in \mathbb{N}_0$, let $F_k\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{(m_k+1)q\times(m_k+1)q}$ be defined by Notation 6.8. For all $k \in \mathbb{Z}_{0,\kappa}$, then there are functions $\Gamma_k\colon \mathbb{C} \setminus \mathbb{R} \to$*

$\mathbb{C}^{(m_k+2)q\times(2m_k+2)q}$ *and* $\Delta_k\colon \mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{(2m_k+2)q\times(m_k+2)q}$ *such that* $P_k^{[f]}(z) = \Gamma_k(z)Q_k^{[f]}(z)\Gamma_k^*(z)$ *and* $Q_k^{[f]}(z) = \Delta_k(z)P_k^{[f]}(z)\Delta_k^*(z)$ *hold true for each* $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof. (I) In the trivial case $k = 0$, choose $\Gamma_0(z) := I_{2q}$ and $\Delta_0(z) := I_{2q}$ for all $z \in \mathbb{C}\setminus\mathbb{R}$.

(II) Now we consider the case that $\kappa \geq 1$ and that $n \in \mathbb{N}_0$ is such that $2n+1 \leq \kappa$. Let

$$\Delta_{2n+1}(z) := \begin{bmatrix} I_{(n+1)q} & 0 \\ [R_{T_{q,n}^*}(z)]^* T_{q,n} & [R_{T_{q,n}^*}(z)]^* v_{q,n} \end{bmatrix} \quad \text{and} \\ \Gamma_{2n+1}(z) := \begin{bmatrix} I_{(n+1)q} & 0 \\ -v_{q,n}^*[R_{T_{q,n}^*}(z)]^* T_{q,n} & v_{q,n}^* \end{bmatrix} \tag{39}$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$. Since $s_j^* = s_j$ holds true for each $j \in \mathbb{Z}_{0,\kappa}$, we have $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. We consider an arbitrary $z \in \mathbb{C}\setminus\mathbb{R}$. Let

$$B_{2n+1}(z) := R_{T_{q,n}}(z)[v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})], \tag{40}$$

let

$$C_{2n+1}(z) := \frac{(z-\alpha)f(z) - [(z-\alpha)f(z)]^*}{z-\overline{z}}, \tag{41}$$

and let

$$\Delta_{2n+1}(z)P_{2n+1}^{[f]}(z)\Delta_{2n+1}^*(z) = \begin{bmatrix} X_{2n+1}(z) & Y_{2n+1}(z) \\ Z_{2n+1}(z) & W_{2n+1}(z) \end{bmatrix} \tag{42}$$

be the $(n+1)q \times (n+1)q$ block representation of $\Delta_{2n+1}(z)P_{2n+1}^{[f]}(z)\Delta_{2n+1}^*(z)$. Then

$$P_{2n+1}^{[f]}(z) = \begin{bmatrix} H_{\alpha\triangleright n} & B_{2n+1}(z) \\ B_{2n+1}^*(z) & C_{2n+1}(z) \end{bmatrix}.$$

Consequently, using (42) and (39), straightforward calculations show that

$$X_{2n+1}(z) = H_{\alpha\triangleright n}, \\ Y_{2n+1}(z) = H_{\alpha\triangleright n}T_{q,n}^* R_{T_{q,n}^*}(z) + B_{2n+1}(z)v_{q,n}^* R_{T_{q,n}^*}(z), \tag{43}$$

$$Z_{2n+1}(z) = \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}H_{\alpha\triangleright n} + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}B_{2n+1}^*(z), \tag{44}$$

and

$$\begin{aligned} W_{2n+1}(z) = {} & \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}H_{\alpha\triangleright n}T_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}B_{2n+1}(z)v_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}B_{2n+1}^*(z)T_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}C_{2n+1}(z)v_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{45}$$

hold true. Because of (44), (40), and (37), we see that

$$\begin{aligned} Y_{2n+1}(z) &= H_{\alpha\triangleright n}T_{q,n}^* R_{T_{q,n}^*}(z) + R_{T_{q,n}}(z)\big[v_{q,n}(z-\alpha)f(z) - \\ &\quad - (-\alpha u_n - y_{0,n})\big]v_{q,n}^* R_{T_{q,n}^*}(z) = \\ &= F_{2n+1}(z) \end{aligned} \tag{46}$$

is valid. From (44), $H^*_{\alpha\rhd n} = H_{\alpha\rhd n}$, (43), and (46) we obtain then

$$Z_{2n+1}(z) = Y^*_{2n+1}(z) = F^*_{2n+1}(z). \tag{47}$$

Using (45), it follows

$$\begin{aligned} W_{2n+1}(z) = {} & \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\rhd n} T^*_{q,n} R_{T^*_{q,n}}(z) + \\ & + \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ & + \Big(\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\Big)^* + \\ & + \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} C_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z). \end{aligned} \tag{48}$$

In view of Lemma 6.7, we have

$$\begin{aligned} & (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\rhd n} T^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = H_{\alpha\rhd n} T^*_{q,n} R_{T^*_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\rhd n} + \\ & \quad + \Big[R_{T^*_{q,n}}(z)\Big]^* \big[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v^*_{q,n}\big] R_{T^*_{q,n}}(z). \end{aligned} \tag{49}$$

By virtue of (40), Remark 6.6, and (37), we conclude

$$\begin{aligned} & (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) = (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* \times \\ & \quad \times T_{q,n} R_{T_{q,n}}(z)[v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = \Big(R_{T_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^*\Big) \times \\ & \quad \times [v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = R_{T_{q,n}}(z)[v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* [v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = F_{2n+1}(z) - H_{\alpha\rhd n} T^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* [v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = F_{2n+1}(z) - H_{\alpha\rhd n} T^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n}(z-\alpha)f(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ & \quad + \Big[R_{T^*_{q,n}}(z)\Big]^* (-\alpha u_n - y_{0,n}) v^*_{q,n} R_{T^*_{q,n}}(z), \end{aligned} \tag{50}$$

which implies

$$\begin{aligned} & (z-\overline{z})\Big(\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\Big)^* = \\ & = -F^*_{2n+1}(z) + \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H^*_{\alpha\rhd n} + \end{aligned} \tag{51}$$

$$+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v^*_{q,n} R_{T^*_{q,n}}(z) -$$
$$- \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}(-\alpha u_n - y_{0,n})^* R_{T^*_{q,n}}(z).$$

Taking into account (41) we get

$$\begin{aligned}(z-\overline{z})\left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} C_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) &= \\ = \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}(z-\alpha)f(z) v^*_{q,n} R_{T^*_{q,n}}(z) &- \\ - \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v^*_{q,n} R_{T^*_{q,n}}(z).\end{aligned} \tag{52}$$

In view of (48), we obtain

$$\begin{aligned}(z-\overline{z})W_{2n+1}(z) = (z-\overline{z})&\left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ (z-\overline{z})\left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ (z-\overline{z})\left(\left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\right)^* + \\ &+ (z-\overline{z})\left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} C_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\end{aligned}$$

and, using (49), (50), (51), (52), and $H^*_{\alpha\triangleright n} = H_{\alpha\triangleright n}$, consequently,

$$\begin{aligned}(z-\overline{z})W_{2n+1}(z) &= H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) - \left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H_{\alpha\triangleright n} + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* \left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v^*_{q,n}\right] R_{T^*_{q,n}}(z) + \\ &+ F_{2n+1}(z) - H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) - \\ &- \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}(z-\alpha)f(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* (-\alpha u_n - y_{0,n}) v^*_{q,n} R_{T^*_{q,n}}(z) - F^*_{2n+1}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H^*_{\alpha\triangleright n} + \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &- \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}(-\alpha u_n - y_{0,n})^* R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}(z-\alpha)f(z) v^*_{q,n} R_{T^*_{q,n}}(z) - \\ &- \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v^*_{q,n} R_{T^*_{q,n}}(z) = F_{2n+1}(z) - F^*_{2n+1}(z).\end{aligned} \tag{53}$$

From (42), (43), (46), (47), (53), and (38) we infer

$$\Delta_{2n+1}(z) P^{[f]}_{2n+1}(z) \Delta^*_{2n+1}(z) = Q^{[f]}_{2n+1}(z). \tag{54}$$

In view of $v^*_{q,n}[R^*_{T^*_{q,n}}(z)]v_{q,n} = I_q$, we easily see that the matrices $\Gamma_{2n+1}(z)$ and $\Delta_{2n+1}(z)$ given by (39) obviously fulfill

$$\Gamma_{2n+1}(z)\Delta_{2n+1}(z) = I_{(n+2)q}. \tag{55}$$

Thus, because of (54), we obtain

$$\begin{aligned} P_{2n+1}^{[f]}(z) &= I_{(n+2)q} P_{2n+1}^{[f]}(z) I_{(n+2)q}^* = \\ &= \Gamma_{2n+1}(z)\Delta_{2n+1}(z)P_{2n+1}^{[f]}(z)\Gamma_{2n+1}^*(z)\Delta_{2n+1}^*(z) = \\ &= \Gamma_{2n+1}(z)Q_{2n+1}^{[f]}(z)\Gamma_{2n+1}^*(z). \end{aligned}$$

In this case $k = 2n+1$ with some $n \in \mathbb{N}_0$, the proof is complete.

(III) Now we consider the case that $\kappa \geq 2$ and that there is an $n \in \mathbb{N}$ such that $k = 2n$. Let $\Gamma_{2n} := \Gamma_{2n+1}$ and let $\Delta_{2n} := \Delta_{2n+1}$. We consider again an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$. Let

$$\Delta_{2n}(z)P_{2n}^{[f]}(z)\Delta_{2n}^*(z) = \begin{bmatrix} X_{2n}(z) & Y_{2n}(z) \\ Z_{2n}(z) & W_{2n}(z) \end{bmatrix} \tag{56}$$

be the $(n+1)q \times (n+1)q$ block representation of $\Delta_{2n}(z)P_{2n}^{[f]}(z)\Delta_{2n}^*(z)$. Setting

$$B_{2n}(z) := R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n] \quad \text{and} \quad C_{2n}(z) := \frac{f(z) - f^*(z)}{z - \overline{z}}, \tag{57}$$

we have $P_{2n}^{[f]}(z) = \left[\begin{smallmatrix} H_n & B_{2n}(z) \\ B_{2n}^*(z) & C_{2n}(z) \end{smallmatrix}\right]$. Consequently, from (56) we easily see then that

$$X_{2n}(z) = H_n, \qquad Y_{2n}(z) = H_n T_{q,n}^* R_{T_{q,n}^*}(z) + B_{2n}(z)v_{q,n}^* R_{T_{q,n}^*}(z), \tag{58}$$

$$Z_{2n}(z) = \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}H_n + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}B_{2n}^*(z), \tag{59}$$

and

$$\begin{aligned} W_{2n}(z) = &\left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}H_nT_{q,n}^*R_{T_{q,n}^*}(z) + \\ &+ \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}B_{2n}^*(z)T_{q,n}^*R_{T_{q,n}^*}(z) + \\ &+ \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n}B_{2n}(z)v_{q,n}^*R_{T_{q,n}^*}(z) + \\ &+ \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}C_{2n}(z)v_{q,n}^*R_{T_{q,n}^*}(z) \end{aligned} \tag{60}$$

hold true. Because of (58), (57), and (35), we obtain

$$Y_{2n}(z) = H_nT_{q,n}^*R_{T_{q,n}^*}(z) + R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n]v_{q,n}^*R_{T_{q,n}^*}(z) = F_{2n}(z). \tag{61}$$

Since $s_j^* = s_j$ is supposed for each $j \in \mathbb{Z}_{0,\kappa}$, we get $H_n^* = H_n$. Consequently, in view of (59), (58), and (61), then

$$Z_{2n}(z) = Y_{2n}^*(z) = F_{2n}^*(z) \tag{62}$$

follows. By virtue of (60) and (57), we see that

$$\begin{aligned} W_{2n}(z) = &\left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H_n T^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} \left[f^*(z) v^*_{q,n} - u^*_n\right] \left[R_{T_{q,n}}(z)\right]^* T^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} \left[\frac{f(z) - f^*(z)}{z - \overline{z}}\right] v^*_{q,n} R_{T^*_{q,n}}(z) \end{aligned} \tag{63}$$

holds true. Taking into account (63) and Remark 6.6, we conclude

$$\begin{aligned} W_{2n}(z) = &\left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H_n T^*_{q,n} R_{T^*_{q,n}}(z) + \\ + &\left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} \left[f^*(z) v^*_{q,n} - u^*_n\right] \left[\frac{1}{z - \overline{z}} \left(R_{T^*_{q,n}}(z) - \left[R_{T_{q,n}}(z)\right]^*\right)\right] + \\ + &\left[\frac{1}{z - \overline{z}} \left(R_{T_{q,n}}(z) - \left[R_{T^*_{q,n}}(z)\right]^*\right)\right] [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ + &\left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} \left[\frac{f(z) - f^*(z)}{z - \overline{z}}\right] v^*_{q,n} R_{T^*_{q,n}}(z). \end{aligned} \tag{64}$$

Using Lemma 6.7, the equation $H^*_n = H_n$, (35), and (64), we infer

$$\begin{aligned} W_{2n}(z) = &\frac{1}{z - \overline{z}} \Big\{ H_n T^*_{q,n} R_{T^*_{q,n}}(z) - \left[R_{T^*_{q,n}}(z)\right]^* T_{q,n} H_n + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* (v_{q,n} u^*_n - u_n v^*_{q,n}) R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} \left[f^*(z) v^*_{q,n} - u^*_n\right] \left(R_{T^*_{q,n}}(z) - \left[R_{T_{q,n}}(z)\right]^*\right) + \\ &+ \left(R_{T_{q,n}}(z) - \left[R_{T^*_{q,n}}(z)\right]^*\right) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^* v_{q,n} [f(z) - f^*(z)] v^*_{q,n} R_{T^*_{q,n}}(z) \Big\} = \\ &= \frac{1}{z - \overline{z}} \Big\{ H_n T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) - \\ &- \left(H_n T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z)\right)^* \Big\} = \\ &= \frac{1}{z - \overline{z}} [F_{2n}(z) - F^*_{2n}(z)]. \end{aligned}$$

Thus, (56), the first equation in (58), (61), (62), and (36) show that

$$\Delta_{2n}(z) P^{[f]}_{2n}(z) \Delta^*_{2n}(z) = \begin{bmatrix} H_n & F_{2n}(z) \\ F^*_{2n}(z) & \frac{F_{2n}(z) - F^*_{2n}(z)}{z - \overline{z}} \end{bmatrix} = Q^{[f]}_{2n}(z) \tag{65}$$

is valid. Because of $\Gamma_{2n} = \Gamma_{2n+1}$ and $\Delta_{2n} = \Delta_{2n+1}$, equation (55) implies $\Gamma_{2n}(z)\Delta_{2n}(z) = I_{(n+2)q}$. Consequently, from (65) we get

$$P^{[f]}_{2n}(z) = \Gamma_{2n}(z)\Delta_{2n}(z) P^{[f]}_{2n}(z) \Delta^*_{2n}(z) \Gamma^*_{2n}(z) = \Gamma_{2n}(z) Q^{[f]}_{2n}(z) \Gamma^*_{2n}(z). \qquad \square$$

Remark 6.10. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Let $f\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function. In view of Proposition 6.9 and Lemma 3.3, it is readily checked that the following statements hold true:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. If $P_{2n}^{[f]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $z \in \mathbb{C}\setminus\mathbb{R}$, then $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (35) belongs to $\mathcal{R}'_{0,(n+1)q}(\Pi_+)$ and the matricial spectral measure μ_{2n} of F_{2n} fulfills $\mu_{2n}(\mathbb{R}) \leq H_n$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. If $P_{2n+1}^{[f]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for each $z \in \mathbb{C}\setminus\mathbb{R}$, then $F_{2n+1}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by (37) belongs to $\mathcal{R}'_{0,(n+1)q}(\Pi_+)$ and the matricial spectral measure μ_{2n+1} of F_{2n+1} fulfills $\mu_{2n+1}(\mathbb{R}) \leq H_{\alpha\triangleright n}$.*

Lemma 6.11. *Let $\alpha \in \mathbb{R}$, let $f\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q\times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$, let $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by (35), and let $\Psi_{2n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by*

$$\Psi_{2n}(z) := R_{T_{q,n}}(z)(H_n T_{q,n}^* - u_n v_{q,n}^* - zT_{q,n}H_nT_{q,n}^*)R_{T_{q,n}^*}(z). \tag{66}$$

Then Ψ_{2n} is a continuous matrix-valued function such that $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. In view of (8), furthermore,

$$F_{2n}(z) = \Psi_{2n}(z) + E_{q,n}(z)f(z)E_{q,n}^*(\overline{z}) \qquad \text{for each } z \in \Pi_+. \tag{67}$$

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $F_{2n+1}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by (37). Then $\Psi_{2n+1}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by*

$$\begin{aligned}\Psi_{2n+1}(z) := R_{T_{q,n}}(z)\big[&H_{\alpha\triangleright n}T_{q,n}^* - (-\alpha u_n - y_{0,n})v_{q,n}^* -\\ &- zT_{q,n}H_{\alpha\triangleright n}T_{q,n}^*\big]R_{T_{q,n}^*}(z)\end{aligned} \tag{68}$$

is continuous and fulfills $\Psi_{2n+1}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ as well as

$$F_{2n+1}(z) = \Psi_{2n+1}(z) + E_{q,n}(z)[(z-\alpha)f(z)]E_{q,n}^*(\overline{z}) \qquad \text{for each } z \in \Pi_+.$$

Proof. (a) The case $n = 0$ is trivial. Suppose now $0 < 2n \leq \kappa$. Remark 4.2 shows that Ψ_{2n} is continuous. For each $x \in \mathbb{R}$, we have $R_{T_{q,n}^*}(x) = [R_{T_{q,n}}(\overline{x})]^* = [R_{T_{q,n}}(x)]^*$ and, consequently,

$$[\Psi_{2n}(x)]^* = R_{T_{q,n}}(x)(T_{q,n}H_n - v_{q,n}u_n^* - xT_{q,n}H_n^*T_{q,n}^*)R_{T_{q,n}^*}(x),$$

which, in view of $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$, i. e., $H_n^* = H_n$, implies that

$$\begin{aligned}[\Psi_{2n}(x)]^* &= R_{T_{q,n}}(x)\big(-[H_nT_{q,n}^* - T_{q,n}H_n] +\\ &\quad + H_nT_{q,n}^* - v_{q,n}u_n^* - xT_{q,n}H_nT_{q,n}^*\big)R_{T_{q,n}^*}(x) =\\ &= R_{T_{q,n}}(x)\big(-[u_nv_{q,n}^* - v_{q,n}u_n^*] + H_nT_{q,n}^* - v_{q,n}u_n^* - xT_{q,n}H_nT_{q,n}^*\big)R_{T_{q,n}^*}(x) =\\ &= R_{T_{q,n}}(x)(H_nT_{q,n}^* - u_nv_{q,n}^* - xT_{q,n}H_nT_{q,n}^*)R_{T_{q,n}^*}(x) = \Psi_{2n}(x)\end{aligned}$$

holds true for all $x \in \mathbb{R}$. Hence, $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Taking into account (35), Remark 4.2, and (66), for all $z \in \Pi_+$, we conclude

$$\begin{aligned} F_{2n}(z) &= R_{T_{q,n}}(z)\big[R_{T_{q,n}}(z)\big]^{-1} H_n T_{q,n}^* R_{T_{q,n}^*}(z) + \\ &\quad + R_{T_{q,n}}(z) v_{q,n} f(z) v_{q,n}^* R_{T_{q,n}^*}(z) - R_{T_{q,n}}(z) u_n v_{q,n}^* R_{T_{q,n}^*}(z) = \\ &= R_{T_{q,n}}(z)\big[(I_{(n+1)q} - zT_{q,n}) H_n T_{q,n}^* - u_n v_{q,n}^*\big] R_{T_{q,n}^*}(z) + \\ &\quad + R_{T_{q,n}}(z) v_{q,n} f(z) v_{q,n}^* \big[R_{T_{q,n}}(\overline{z})\big]^* = \\ &= R_{T_{q,n}}(z)(H_n T_{q,n}^* - zT_{q,n} H_n T_{q,n}^* - u_n v_{q,n}^*) R_{T_{q,n}^*}(z) + \\ &\quad + R_{T_{q,n}}(z) v_{q,n} f(z) [R_{T_{q,n}}(\overline{z}) v_{q,n}]^* = \\ &= \Psi_{2n}(z) + E_{q,n}(z) f(z) E_{q,n}^*(\overline{z}). \end{aligned}$$

(b) Part (b) can be proved analogously. We omit the details. □

Lemma 6.12. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Further, let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \quad \text{and} \quad P_{2n+1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \quad \text{for all} \quad z \in \Pi_+. \tag{69}$$

Then the $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\geq,1}^{q}([\alpha,\infty))$.

Proof. (I) For all $z \in \Pi_+$, from Remark 4.4 we see that (11) holds true and, in view of (69), hence, that the block matrix on the left-hand side of (11) is non-negative Hermitian. Consequently, since S is holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$, Lemma 3.3 yields that $F := \operatorname{Rstr}_{\Pi_+} S$ belongs to $\mathcal{R}_{0,q}'(\Pi_+)$ and that the matricial spectral measure μ of F fulfills $\mu(\mathbb{R}) \leq s_0$. Thus, Remark 3.6 provides us $\sigma([\alpha,\infty)) = \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu([\alpha,\infty)) = \mu([\alpha,\infty)) \leq \mu(\mathbb{R}) \leq s_0$. Because of (69) and (9), we have $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. In particular, $s_0 \in \mathbb{C}_{\geq}^{q\times q}$. Hence,

$$s_0^* = s_0 \quad \text{and} \quad \{u^*\sigma([\alpha,\infty))u, u^* s_0 u\} \subseteq [0,\infty) \quad \text{for all } u \in \mathbb{C}^q. \tag{70}$$

(II) In the second part of the proof, we consider an arbitrary $n \in \mathbb{N}$ and an arbitrary $u \in \mathbb{C}^q$. From Remark 3.4 we see then that

$$\int_{[\alpha,\infty)} \left|\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}\right| (u^*\sigma u)(\mathrm{d}t) = n \int_{[\alpha,\infty)} \left|\frac{1}{t-(\mathrm{i}n+\alpha)}\right| (u^*\sigma u)(\mathrm{d}t) < \infty. \tag{71}$$

In view of

$$\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)} = -\frac{n^2}{|t-\alpha-\mathrm{i}n|^2} + \mathrm{i}\frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2} \tag{72}$$

and (71), we obtain

$$\begin{aligned} &\int_{[\alpha,\infty)} \left|-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2}\right| (u^*\sigma u)(\mathrm{d}t) = \\ &= \int_{[\alpha,\infty)} \left|\Re\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}\right]\right| (u^*\sigma u)(\mathrm{d}t) < \infty \end{aligned}$$

and

$$\begin{aligned}\int_{[\alpha,\infty)} \left|\frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2}\right| (u^*\sigma u)(\mathrm{d}t) &= \\ = \int_{[\alpha,\infty)} \left|\Im\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}\right]\right| (u^*\sigma u)(\mathrm{d}t) &< \infty.\end{aligned} \tag{73}$$

For each $t \in [\alpha, \infty)$, we have

$$n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)} + 1\right] = \frac{(t-\alpha)n}{t-\alpha-\mathrm{i}n} = \frac{(t-\alpha)^2 n}{(t-\alpha)^2+n^2} + \mathrm{i}\frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}. \tag{74}$$

Consequently, the function $g_n\colon [\alpha,\infty) \to \mathbb{C}$ given by $g_n(t) := n[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)} + 1]$ fulfills $|\Re[g_n(t)]| = n(t-\alpha)^2[(t-\alpha)^2+n^2]^{-1} \leq n = n \cdot 1_{[\alpha,\infty)}(t)$ and

$$\begin{aligned}|\Im[g_n(t)]| &= (t-\alpha)n^2\left[(t-\alpha)^2+n^2\right]^{-1} \leq \\ &\leq 2|t-\alpha|n^2\left[(t-\alpha)^2+n^2\right]^{-1} \leq n = n\cdot 1_{[\alpha,\infty)}(t)\end{aligned}$$

for each $t \in [\alpha,\infty)$. This implies $\int_{[\alpha,\infty)} |\Re[g_n(t)]|(u^*\sigma u)(\mathrm{d}t) \leq nu^*\sigma([\alpha,\infty))u < \infty$ and $\int_{[\alpha,\infty)} |\Im[g_n(t)]|(u^*\sigma u)(\mathrm{d}t) \leq nu^*\sigma([\alpha,\infty))u < \infty$. Thus,

$$g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, u^*\sigma u; \mathbb{C}). \tag{75}$$

Using Theorem 3.5, Remark 7.1, (72), and (73), we conclude

$$\begin{aligned}u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u &= u^*\left(\mathrm{i}n\int_{[\alpha,\infty)}[t-(\mathrm{i}n+\alpha)]^{-1}\sigma(\mathrm{d}t)\right)u = \\ &= \int_{[\alpha,\infty)} \frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}(u^*\sigma u)(\mathrm{d}t) = \\ &= \int_{[\alpha,\infty)}\left[-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2} + \mathrm{i}\frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2}\right](u^*\sigma u)(\mathrm{d}t) = \\ &= -n^2\int_{[\alpha,\infty)}\frac{1}{|t-\alpha-\mathrm{i}n|^2}(u^*\sigma u)(\mathrm{d}t) + \mathrm{i}n\int_{[\alpha,\infty)}\frac{t-\alpha}{|t-\alpha-\mathrm{i}n|^2}(u^*\sigma u)(\mathrm{d}t)\end{aligned}$$

and, in particular,

$$\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) = -n^2\int_{[\alpha,\infty)}|t-\alpha-\mathrm{i}n|^{-2}(u^*\sigma u)(\mathrm{d}t). \tag{76}$$

Taking into account $\sigma([\alpha,\infty)) \leq s_0$, (76), and that $1 - n^2|t-\alpha-\mathrm{i}n|^{-1} = (t-\alpha)^2[(t-\alpha)^2+n^2]^{-1}$ holds true, for each $t\in[\alpha,\infty)$, we get

$$\begin{aligned}&\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*s_0u \geq \Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*\sigma([\alpha,\infty))u \\ &= \int_{[\alpha,\infty)}\left(1-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2}\right)(u^*\sigma u)(\mathrm{d}t) = \int_{[\alpha,\infty)}\frac{(t-\alpha)^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \geq 0\end{aligned}$$

and, consequently,

$$\begin{aligned}&[\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*s_0u]^2 \geq \\ &\geq [\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*\sigma([\alpha,\infty))u]^2.\end{aligned} \tag{77}$$

Because of (70), (77), and again (70), it follows

$$\begin{aligned}&|nu^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0]u|^2=n^2|u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u+u^*s_0u|^2=\\&=n^2\Big([\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u)+u^*s_0u]^2+[\Im(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u)]^2\Big)\geq\\&\geq n^2\Big([\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u)+u^*\sigma([\alpha,\infty))u]^2+[\Im(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u)]^2\Big)=\\&=n^2|u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u+u^*\sigma([\alpha,\infty))u|^2=\\&=|nu^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+\sigma([\alpha,\infty))]u|^2\end{aligned}$$

and, therefore,

$$|nu^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0]u|\geq|nu^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+\sigma([\alpha,\infty))]u|. \tag{78}$$

Since S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$, the function $G\colon\Pi_+\to\mathbb{C}^{q\times q}$ given by $G(w):=wS(w+\alpha)+s_0$ is holomorphic in Π_+. From Remark 4.4 we know that, for all $z\in\mathbb{C}\setminus[\alpha,\infty)$, equation (12) is true. Hence, from (69) we see that the block matrix on the left-hand side of (12) is non-negative Hermitian. Consequently, we conclude

$$\begin{aligned}&\begin{bmatrix}-\alpha s_0+s_1 & G(w)\\ G^*(w) & \frac{G(w)-G^*(w)}{w-\overline{w}}\end{bmatrix}=\begin{bmatrix}-\alpha s_0+s_1 & wS(w+\alpha)+s_0\\ [wS(w+\alpha)+s_0]^* & \frac{[wS(w+\alpha)+s_0]-[wS(w+\alpha)+s_0]^*}{w-\overline{w}}\end{bmatrix}=\\&=\begin{bmatrix}-\alpha s_0+s_1 & [(w+\alpha)-\alpha]S(w+\alpha)+s_0\\ ([(w+\alpha)-\alpha]S(w+\alpha)+s_0)^* & \frac{[(w+\alpha)-\alpha]S(w+\alpha)-([(w+\alpha)-\alpha]S(w+\alpha))^*}{w-\overline{w}}\end{bmatrix}\in\mathbb{C}^{2q\times 2q}_{\geq}.\end{aligned}\tag{79}$$

Since G is holomorphic, from (79) and Lemma 3.3 then $\sup_{y\in(0,\infty)}(y\|G(\mathrm{i}y)\|_{\mathrm{S}})\leq\|-\alpha s_0+s_1\|_{\mathrm{S}}$ and, hence, $\sup_{n\in\mathbb{N}}(n\|\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0\|_{\mathrm{S}})\leq\|-\alpha s_0+s_1\|_{\mathrm{S}}$ follows. Thus, the Bunjakowski–Cauchy–Schwarz inequality provides us

$$\begin{aligned}&|u^*(n[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0])u|\leq\|n[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0]u\|_{\mathrm{E}}\cdot\|u\|_{\mathrm{E}}\leq\\&\leq n\|\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0\|_{\mathrm{S}}\cdot\|u\|_{\mathrm{E}}^2\leq\|-\alpha s_0+s_1\|_{\mathrm{S}}\cdot\|u\|_{\mathrm{E}}^2.\end{aligned}\tag{80}$$

For each $t\in[\alpha,\infty)$, we have $|t-\alpha|=\liminf_{n\to\infty}(t-\alpha)n^2[(t-\alpha)^2+n^2]^{-1}$. Then

$$\begin{aligned}\int_{[\alpha,\infty)}|t-\alpha|(u^*\sigma u)(\mathrm{d}t)&=\int_{[\alpha,\infty)}\liminf_{n\to\infty}\frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t)\leq\\&\leq\liminf_{n\to\infty}\int_{[\alpha,\infty)}\frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t),\end{aligned}\tag{81}$$

by virtue of Fatou's lemma. Obviously, from (75) and (74) we infer

$$\begin{aligned}&\int_{[\alpha,\infty)}\Im\left(n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\right)(u^*\sigma u)(\mathrm{d}t)=\\&=\int_{[\alpha,\infty)}\frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t)\end{aligned}\tag{82}$$

and

$$\begin{aligned}\int_{[\alpha,\infty)} \Im\left(n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\right)(u^*\sigma u)(\mathrm{d}t) = \\ = \Im\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t)\right) \leq \\ \leq \left|\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t)\right|.\end{aligned} \tag{83}$$

(III) Since (75) holds true for every choice of $u \in \mathbb{C}^q$ and $n \in \mathbb{N}$, Remark 7.1 yields $g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$. Hence, Remark 7.1 shows that

$$\begin{aligned}\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t) = \\ = u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\end{aligned} \tag{84}$$

is valid for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$. Combining (82), (83), and (84), we have

$$\begin{aligned}0 \leq \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \leq \\ \leq \left|u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\right|\end{aligned} \tag{85}$$

for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and all $t \in [\alpha,\infty)$, we see that $g_n(t) - n \cdot 1_{[\alpha,\infty)}(t) = \tilde{g}_n(t)$ holds true, where $\tilde{g}_n \colon [\alpha,\infty) \to \mathbb{C}$ is given by $\tilde{g}_n(t) := \mathrm{i}n^2[t-(\mathrm{i}n+\alpha)]^{-1}$. Thus, for each $n \in \mathbb{N}$, we get $\tilde{g}_n = g_n - n \cdot 1_{[\alpha,\infty)}$, and, since $g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$, then $\tilde{g}_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ and

$$\int_{[\alpha,\infty)} \tilde{g}_n \mathrm{d}\sigma = \int_{[\alpha,\infty)} g_n \mathrm{d}\sigma - n\int_{[\alpha,\infty)} 1_{[\alpha,\infty)} \mathrm{d}\sigma = \int_{[\alpha,\infty)} g_n \mathrm{d}\sigma - n\sigma([\alpha,\infty))$$

hold true as well. Consequently, for each $n \in \mathbb{N}$, we conclude

$$\begin{aligned}\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{\mathrm{i}n^2}{t-(\mathrm{i}n+\alpha)}\sigma(\mathrm{d}t) + n\sigma([\alpha,\infty)) = \\ = \mathrm{i}n^2\int_{[\alpha,\infty)} \frac{1}{t-(\mathrm{i}n+\alpha)}\sigma(\mathrm{d}t) + n\sigma([\alpha,\infty)) = n[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha) + \sigma([\alpha,\infty))].\end{aligned}$$

Thus, because of (78), for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$, we obtain

$$\left|u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\right| \leq |nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0]u|. \tag{86}$$

Taking into account (81), (85), (86), and (80), for each $u \in \mathbb{C}^q$, we get

$$\begin{aligned}\int_{[\alpha,\infty)} |t-\alpha|(u^*\sigma u)(\mathrm{d}t) &\le \liminf_{n\to\infty} \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \le \\ &\le \liminf_{n\to\infty} \left| u^* \left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right) u \right| \le \\ &\le \liminf_{n\to\infty} |nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0]u| \le \\ &\le \liminf_{n\to\infty} \|-\alpha s_0 + s_1\|_{\mathrm{S}} \cdot \|u\|_{\mathrm{E}}^2 = \|-\alpha s_0 + s_1\|_{\mathrm{S}} \cdot \|u\|_{\mathrm{E}}^2 < \infty.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\int_{[\alpha,\infty)} |t|(u^*\sigma u)(\mathrm{d}t) &\le \int_{[\alpha,\infty)} (|t-\alpha|+|\alpha|)(u^*\sigma u)(\mathrm{d}t) = \\ &= \int_{[\alpha,\infty)} |t-\alpha|(u^*\sigma u)(\mathrm{d}t) + \int_{[\alpha,\infty)} |\alpha|(u^*\sigma u)(\mathrm{d}t) \le \\ &\le \|-\alpha s_0 + s_1\|_{\mathrm{S}} \cdot \|u\|_{\mathrm{E}}^2 + |\alpha|(u^*\sigma u)([\alpha,\infty)) < \infty\end{aligned}$$

is true for all $u \in \mathbb{C}^q$. Thus, Remark 7.1 provides us $\sigma \in \mathcal{M}_{\ge,1}^q([\alpha,\infty))$. □

Lemma 6.13. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \le \kappa$. Further, let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $P_{2n}^{[S]}(z) \in \mathbb{C}_{\ge}^{(n+2)q\times(n+2)q}$ and $P_{2n+1}^{[S]}(z) \in \mathbb{C}_{\ge}^{(n+2)q\times(n+2)q}$ hold true for all $z \in \Pi_+$. Then:*

(a) *The $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\ge,1}^q([\alpha,\infty))$.*
(b) *The function $\phi\colon [\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\phi(t) := \sqrt{t-\alpha}I_q$ belongs to $q \times q$-$\mathcal{L}^2([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ and $\sigma^{\#}\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ defined by (14) belongs to $\mathcal{M}_{\ge}^q([\alpha,\infty))$.*
(c) *The function $\tilde{S}\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\tilde{S}(z) := (z-\alpha)S(z)$ and the $[\alpha,\infty)$-Stieltjes transform $S^{[\sigma^{\#}]}$ of $\sigma^{\#}$ fulfill $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.*
(d) *The function $(\tilde{S})_\square := \mathrm{Rstr}_{\Pi_+} \tilde{S}$ belongs to $\mathcal{R}_q'(\Pi_+)$ and $(\tilde{\sigma})_\square\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}$ given by $(\tilde{\sigma})_\square(B) := \sigma^{\#}(B\cap[\alpha,\infty))$ is exactly the matricial spectral measure of $(\tilde{S})_\square$.*

Proof. (a) Part (a) is proved in Lemma 6.12.

(b) In view of (a), part (b) follows immediately from Remark 4.8.

(c) Let $z \in \mathbb{C}\setminus[\alpha,\infty)$. According to Remark 3.4 and Theorem 3.5, the function $g_{\alpha,z}\colon [\alpha,\infty) \to \mathbb{C}$ given by $g_{\alpha,z}(t) := (z-\alpha)/(t-z)$ belongs to $\mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ and

$$(z-\alpha)S(z) = \int_{[\alpha,\infty)} \frac{z-\alpha}{t-z}\sigma(\mathrm{d}t)$$

is true. Consequently, in view of Lemma 7.2, we get that the pair $[g_{\alpha,z}I_q, 1_{[\alpha,\infty)}I_q]$ is left-integrable with respect to σ and that

$$\tilde{S}(z) = (z-\alpha)S(z) = \int_{[\alpha,\infty)} (\frac{z-\alpha}{t-z}I_q)\sigma(\mathrm{d}t)I_q^*.$$

Due to Remark 7.3, then the pair $[g_{\alpha,z}I_q + 1_{[\alpha,\infty)}I_q, I_q]$ is left-integrable with respect to σ and

$$\int_{[\alpha,\infty)} \left[\left(\frac{z-\alpha}{t-z}+1\right)I_q\right]\sigma(\mathrm{d}t)I_q^* = \int_{[\alpha,\infty)} \left(\frac{z-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* + \int_{[\alpha,\infty)} I_q\sigma(\mathrm{d}t)I_q^*$$

is fulfilled. Taking into account

$$\sigma([\alpha,\infty)) = \int_{[\alpha,\infty)} 1_{[\alpha,\infty)}\mathrm{d}\sigma = \int_{[\alpha,\infty)} (1_{[\alpha,\infty)}I_q)\mathrm{d}\sigma(1_{[\alpha,\infty)}I_q)^* = \int_{[\alpha,\infty)} I_q\sigma(\mathrm{d}t)I_q^*$$

and that $(z-\alpha)/(t-z)+1 = (t-\alpha)/(t-z)$ holds true for each $t \in [\alpha,\infty)$, we get then

$$\tilde{S}(z) = \int_{[\alpha,\infty)} \left(\frac{t-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* - \sigma([\alpha,\infty)). \tag{87}$$

Because of Lemma 7.2, Proposition 7.4, and (14), we have

$$\begin{aligned}
&\int_{[\alpha,\infty)} \left(\frac{t-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* = \\
&= \int_{[\alpha,\infty)} \left[\left(\frac{1}{t-z}I_q\right)(\sqrt{t-\alpha}I_q)\right]\sigma(\mathrm{d}t)\left[I_q(\sqrt{t-\alpha}I_q)\right]^* = \\
&= \int_{[\alpha,\infty)} \left(\frac{1}{t-z}I_q\right)\sigma^{\#}(\mathrm{d}t)I_q^* = \int_{[\alpha,\infty)} \frac{1}{t-z}\sigma^{\#}(\mathrm{d}t) = S^{[\sigma^{\#}]}(z).
\end{aligned}$$

Thus, from (87) it follows $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.

(d) In view of Theorem 3.5, we have $S^{[\sigma^{\#}]} \in \mathcal{S}_{0,q;[\alpha,\infty)}$. Thus, Remark 3.6 shows that $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]} \in \mathcal{R}'_{0,q}(\Pi_+) \subseteq \mathcal{R}'_q(\Pi_+)$, that the matricial spectral measure $\mu^{\#}$ of $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$ fulfills $\sigma^{\#} = \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu^{\#}$, and that $\mu^{\#}(\mathbb{R}\setminus[\alpha,\infty)) = \mu^{\#}((-\infty,\alpha)) = 0_{q\times q}$. Consequently, $(\tilde{\sigma})_{\square}$ is the matricial spectral measure of $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$. From Theorem 3.1 one can see that $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ given by $F(z) := -\sigma([\alpha,\infty))$ belongs to $\mathcal{R}'_q(\Pi_+)$ and that the matricial spectral measure θ of F fulfills $\theta(B) = 0_{q\times q}$ for all $B \in \mathfrak{B}_{\mathbb{R}}$ (see also [13, Beispiel 1.2.1]). Since $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ is valid for all $z \in \mathbb{C}\setminus[\alpha,\infty)$, we get $(\tilde{S})_{\square} = \mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]} + F$. Since $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$ and F both belong to $\mathcal{R}'_q(\Pi_+)$, from [26, Remark 4.4] we see that $(\tilde{S})_{\square} \in \mathcal{R}'_q(\Pi_+)$ and that $(\tilde{\sigma})_{\square} + \theta$ is the matricial spectral measure of $(\tilde{S})_{\square}$. In view of $(\tilde{\sigma})_{\square} + \theta = (\tilde{\sigma})_{\square}$, the proof is complete. $\square$

Lemma 6.14. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \qquad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}. \tag{88}$$

Then the $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\geq,2n}^{q}([\alpha,\infty))$ and the inequality $H_n^{[\sigma]} \leq H_n$ holds true.

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$\left\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\right\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \qquad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}. \tag{89}$$

Then the $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\geq,2n+1}^{q}([\alpha,\infty))$ and the inequality $H_{\alpha\triangleright n}^{[\sigma]} \leq H_{\alpha\triangleright n}$ holds true.

Proof. (a) Because of (88), we get $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q} \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ and, in particular, $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$. In view of $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$, we see that the function S is holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$ and, using additionally [26, Propositions 8.9 and 8.8], we also obtain $\operatorname{Rstr}_{\Pi_+} S \in \mathcal{R}'_{0,q}(\Pi_+) \subseteq \mathcal{R}'_q(\Pi_+)$. Let $f := S$ and let $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by (35). Using Remark 6.10 and [26, Propositions 8.9 and 8.8], we conclude that $F_{2n} \in \mathcal{R}'_{0,(n+1)q}(\Pi_+) \subseteq \mathcal{R}'_{(n+1)q}(\Pi_+)$ and that the matricial spectral measure μ_{2n} of F_{2n} fulfills $\mu_{2n}(\mathbb{R}) \leq H_n$. Let $\Psi_{2n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given (66). Since $s_j^* = s_j$ holds true for each $j \in \mathbb{Z}_{0,2n}$, from Lemma 6.11 we see that Ψ_{2n} is a continuous matrix-valued function with $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Furthermore, Lemma 6.11 yields (67). According to Remark 3.6, the matricial spectral measure $\sigma_\square$ of $\operatorname{Rstr}_{\Pi_+} S$ fulfills $\sigma = \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \sigma_\square$ and $\sigma_\square(\mathbb{R}\setminus[\alpha,\infty)) = 0$. Standard arguments of measure theory show that we can choose sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ of real numbers such that

$$\sigma_\square(\{a_k\}) = 0, \quad \sigma_\square(\{b_k\}) = 0, \quad \mu_{2n}(\{a_k\}) = 0, \quad \mu_{2n}(\{b_k\}) = 0, \tag{90}$$
$$a_k < b_k, \quad \text{and} \quad (a_k, b_k) \subseteq (a_{k+1}, b_{k+1}) \tag{91}$$

hold true for each $k \in \mathbb{N}$ and that $\bigcup_{k=1}^\infty (a_k, b_k) = \mathbb{R}$. In view of $F_{2n} \in \mathcal{R}'_{(n+1)q}(\Pi_+)$, a matricial version of Stieltjes' inversion formula (see [14, Theorem 8.6]), and (90) provide us

$$\begin{aligned}\mu_{2n}((a_k,b_k)) &= \frac{1}{2}[\mu_{2n}(\{a_k\}) + \mu_{2n}(\{b_k\})] + \mu_{2n}((a_k,b_k)) = \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0+0}\int_{[a_k,b_k]} \Im F_{2n}(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x)\end{aligned} \tag{92}$$

for all $k \in \mathbb{N}$, where $\lambda^{(1)}$ is the Lebesgue measure defined on $\mathfrak{B}_\mathbb{R}$. The function $E_{q,n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ given by (8) is holomorphic in $\mathbb{C}$. Moreover, Ψ_{2n} is continuous with $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Thus, for all $k \in \mathbb{N}$, we get from (67), a matricial version of Stieltjes' inversion formula (see [14, Theorem 8.6]) and (90) that

$$\begin{aligned}\frac{1}{\pi}\lim_{\epsilon\to0+0}\int_{[a_k,b_k]}&\Im F_{2n}(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x)=\\=\frac{1}{2}(E_{q,n}(a_k)\sigma_\square(\{a_k\})&[E_{q,n}(a_k)]^*+E_{q,n}(b_k)\sigma_\square(\{b_k\})[E_{q,n}(b_k)]^*)+\\&+\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)=\\=\int_{(a_k,b_k)}&E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t).\end{aligned}\tag{93}$$

Combining (93) and (92), we obtain

$$\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)=\mu_{2n}((a_k,b_k))\leq\mu_{2n}(\mathbb{R})\quad\text{for all }k\in\mathbb{N}\tag{94}$$

and, consequently,

$$\operatorname{tr}\left[\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)\right]\leq\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty\qquad\text{for all }k\in\mathbb{N}.\tag{95}$$

The trace measure $\tau:=\operatorname{tr}\sigma_\square$ of $\sigma_\square$ is a finite measure and $\sigma_\square$ is absolutely continuous with respect to τ. We can choose a version $(\sigma_\square)'_\tau$ of the matricial Radon–Nikodym derivative of $\sigma_\square$ with respect to τ such that $(\sigma_\square)'_\tau(t)\in\mathbb{C}^{q\times q}_{\geq}$ for all $t\in\mathbb{R}$. For all $k\in\mathbb{N}$, then

$$g_k:=\|1_{(a_k,b_k)}(\operatorname{Rstr}_\mathbb{R}E_{q,n})\sqrt{(\sigma_\square)'_\tau}\|^2_\mathrm{F}\in\mathcal{L}^1(\mathbb{R},\mathfrak{B}_\mathbb{R},\tau;\mathbb{C})$$

and

$$\operatorname{tr}[\int_\mathbb{R}(1_{(a_k,b_k)}\operatorname{Rstr}_\mathbb{R}E_{q,n})\mathrm{d}\sigma_\square(1_{(a_k,b_k)}\operatorname{Rstr}_\mathbb{R}E_{q,n})^*]=\int_\mathbb{R}g_k\mathrm{d}\tau.$$

Thus, by virtue of (95), then

$$\int_\mathbb{R}g_k\mathrm{d}\tau\leq\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty\tag{96}$$

follows for all $k\in\mathbb{N}$. Obviously, $g\colon\mathbb{R}\to\mathbb{C}$ defined by $g(t):=\|E_{q,n}(t)\sqrt{(\sigma_\square)'_\tau}\|^2_\mathrm{F}$ is an $\mathfrak{B}_\mathbb{R}$-$\mathfrak{B}_\mathbb{C}$-measurable function with $g(\mathbb{R})\subseteq[0,\infty)$. For all $t\in\mathbb{R}$, we see that

$$\begin{aligned}g(t)&=\left\|\left[\lim_{k\to\infty}1_{(a_k,b_k)}(t)\right]\cdot\left[(\operatorname{Rstr}_\mathbb{R}E_{q,n})\sqrt{(\sigma_\square)'_\tau}\right](t)\right\|^2_\mathrm{F}=\\&=\lim_{k\to\infty}g_k(t)=\liminf_{k\to\infty}g_k(t).\end{aligned}\tag{97}$$

In view of (97) and (96), Fatou's lemma yields then

$$\int_\mathbb{R}|g(t)|\tau(\mathrm{d}t)=\int_\mathbb{R}\liminf_{k\to\infty}g_k(t)\tau(\mathrm{d}t)\leq\liminf_{k\to\infty}\int_\mathbb{R}g_k(t)\tau(\mathrm{d}t)\leq\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty,$$

and, consequently, $g\in\mathcal{L}^1(\mathbb{R},\mathfrak{B}_\mathbb{R},\tau;\mathbb{C})$. Because of Lemma 7.2, we get then $\operatorname{Rstr}_\mathbb{R}E_{q,n}\in(n+1)q\times q$-$\mathcal{L}^2(\mathbb{R},\mathfrak{B}_\mathbb{R},\sigma_\square;\mathbb{C})$. Hence, from $\sigma=\operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}}\sigma_\square$

we obtain that $\operatorname{Rstr}_{[\alpha,\infty)} E_{q,n}$ belongs to $(n+1)q \times q$-$\mathcal{L}^2([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$, that

$$\int_{\mathbb{R}} E_{q,n}(t)\sigma_{\square}(\mathrm{d}t)E_{q,n}^*(t) = \int_{[\alpha,\infty)} E_{q,n}(t)\sigma(\mathrm{d}t)E_{q,n}^*(t), \tag{98}$$

and that $\Theta_n\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by

$$\Theta_n(B) := \int_B E_{q,n}(t)\sigma_{\square}(\mathrm{d}t)E_{q,n}^*(t) \tag{99}$$

is a well-defined non-negative Hermitian $(n+1)q \times (n+1)q$ measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$. Furthermore, applying Remark 4.7, we get $\sigma \in \mathcal{M}_{\geq,2n}^q([\alpha,\infty))$ and (13). Using (99), $\bigcup_{k=1}^{\infty}(a_k, b_k) = \mathbb{R}$, (91), $\Theta_n \in \mathcal{M}_{\geq}^{(n+1)q}(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$, (94), and $\mu_{2n} \in \mathcal{M}_{\geq}^{(n+1)q}(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$, we conclude

$$\int_{\mathbb{R}} E_{q,n}(t)\sigma_{\square}(\mathrm{d}t)E_{q,n}^*(t) = \Theta_n(\mathbb{R}) = \lim_{k\to\infty} \Theta_n((a_k, b_k)) =$$
$$= \lim_{k\to\infty} \int_{(a_k,b_k)} E_{q,n}(t)\sigma_{\square}(\mathrm{d}t)E_{q,n}^*(t) = \lim_{k\to\infty} \mu_{2n}((a_k, b_k)) = \mu_{2n}(\mathbb{R}). \tag{100}$$

The combination of (13), (98), (100), and $\mu_{2n}(\mathbb{R}) \leq H_n$ provides us then

$$H_n^{[\sigma]} = \int_{[\alpha,\infty)} E_{q,n}(t)\sigma(\mathrm{d}t)E_{q,n}^*(t) =$$
$$= \int_{\mathbb{R}} \operatorname{Rstr}_{\mathbb{R}} E_{q,n} \mathrm{d}\sigma_{\square} (\operatorname{Rstr}_{\mathbb{R}} E_{q,n})^* = \mu_{2n}(\mathbb{R}) \leq H_n.$$

(b) Because of (89), we have $\{H_n, H_{\alpha\triangleright n}\} \subseteq \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q} \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ and, consequently, $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n+1}$. Since S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$, from (89) and Lemma 6.13 we infer that σ belongs to $\mathcal{M}_{\geq,1}^q([\alpha,\infty))$, that $\sigma^{\#}\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ defined by (14) belongs to $\mathcal{M}_{\geq}^q([\alpha,\infty))$, that $\tilde{S}\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\tilde{S}(z) := (z-\alpha)S(z)$ is a function with $\operatorname{Rstr}_{\Pi_+} \tilde{S} \in \mathcal{R}_q'(\Pi_+)$, and that $(\sigma^{\#})_{\square}\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}$ given by $(\tilde{\sigma})_{\square}(B) := \sigma^{\#}(B \cap [\alpha,\infty))$ is the matricial spectral measure of $(\tilde{S})_{\square} := \operatorname{Rstr}_{\Pi_+} \tilde{S}$. Observe that Remark 4.8(b) shows that (15) holds true. Now part (b) can be proved analogous to part (a), where $F_{2n+1}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (37) and $\Psi_{2n+1}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by (68) play the roles of F_{2n} and Ψ_{2n}, respectively (for details, see also [48, Lemma 7.9]). □

Remark 6.15. *It is readily checked that if E is non-negative Hermitian, then $\|B\|_{\mathrm{S}}^2 \leq \|A\|_{\mathrm{S}} \cdot \|D\|_{\mathrm{S}}$ (see, e. g. [16, proof of Lemma 1.1.10]).*

Remark 6.16. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Using Remark 6.15 and the definition of the class $\mathcal{S}_{0,q;[\alpha,\infty)}$, it is readily checked that the following statements hold true:*

(a) *If $n \in \mathbb{N}_0$ is such that $2n \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is such that $P_{2n}^{[S]}(\mathrm{i}y) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $y \in (0,\infty)$, then*

$$\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}S(\mathrm{i}y) - u_n] = 0. \tag{101}$$

(b) *If $\kappa \geq 1$ and $n \in \mathbb{N}_0$ are such that $2n+1 \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is such that $P_{2n+1}^{[S]}(\mathrm{i}y) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $y \in (0,\infty)$, then*

$$\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}(\mathrm{i}y-\alpha)S(\mathrm{i}y) - (-\alpha u_n - y_{0,n})] = 0.$$

Remark 6.17. *Let $n \in \mathbb{N}_0$ and let $y \in \mathbb{R}$. If $u \in \mathbb{C}^{(n+1)q\times p}$ is such that $\lim_{y\to\infty}[u^* R_{T_{q,n}}(\mathrm{i}y)u] = 0$, then from Remark 4.2 one can easily see that $u = 0$ holds true.*

Remark 6.18. *Let $n \in \mathbb{N}$ and let $(d_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. If $d_0 = 0_{q\times q}$ and if the block Hankel matrix $[d_{j+k}]_{j,k=0}^n$ is non-negative Hermitian, then a characterization of non-negative Hermitian block matrices by their blocks (see [3, 23], or [16, Lemmata 1.1.9 and 1.1.7]), it is readily proved by induction that $d_j = 0_{q\times q}$ for all $j \in \mathbb{Z}_{0,2n-1}$.*

Lemma 6.19. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for all $z \in \mathbb{C}\setminus\mathbb{R}$. Then the $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\geq,2n}^q([\alpha,\infty))$ and S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $z \in \mathbb{C}\setminus\mathbb{R}$. Then the $[\alpha,\infty)$-Stieltjes measure σ of S belongs to $\mathcal{M}_{\geq,2n+1}^q([\alpha,\infty))$ and S belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.*

Proof. (a) Lemma 6.14 yields $\sigma \in \mathcal{M}_{\geq,2n}^q([\alpha,\infty))$ and $H_n^{[\sigma]} \leq H_n$. If $n = 0$, then $\sigma \in \mathcal{M}_{\geq}^q[[\alpha,\infty); (s_j)_{j=0}^{2n}, \leq]$ follows. Suppose now $n \geq 1$. Remark 6.16 shows that (101) is valid. Obviously, $\sigma \in \mathcal{M}_{\geq}^q[[\alpha,\infty); (s_j^{[\sigma]})_{j=0}^{2n}, \leq]$, where $(s_j^{[\sigma]})_{j=0}^{2n}$ is defined by (1). Thus, Proposition 4.9 and Remark 6.16 provide us $\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}S(\mathrm{i}y) - u_n^{[\sigma]}] = 0$ where $s_{-1}^{[\sigma]} := 0_{q\times q}$ and where $u_n^{[\sigma]} := -\operatorname{col}(s_{j-1}^{[\sigma]})_{j=0}^n$. Using additionally (101), we can conclude

$$\lim_{y\to\infty} (u_n^{[\sigma]} - u_n)^* R_{T_{q,n}}(\mathrm{i}y)(u_n^{[\sigma]} - u_n) = 0.$$

Consequently, Remark 6.17 yields $u_n^{[\sigma]} = u_n$. Let $d_j := s_j - s_j^{[\sigma]}$ for each $j \in \mathbb{Z}_{0,2n}$. Then $u_n^{[\sigma]} = u_n$ and $n \geq 1$ imply $d_0 = 0_{q\times q}$. Furthermore, the inequality $H_n^{[\sigma]} \leq H_n$ shows that the block Hankel matrix $[d_{j+k}]_{j,k=0}^n$ is non-negative Hermitian. Thus, $d_{2n} \in \mathbb{C}_{\geq}^{q\times q}$ and Remark 6.18 yield $d_j = 0_{q\times q}$ for each $j \in \mathbb{Z}_{0,2n-1}$. Hence, σ belongs to $\mathcal{M}_{\geq}^q[[\alpha,\infty); (s_j)_{j=0}^{2n}, \leq]$.

(b) Part (b) can be proved analogous to part (a). We omit the details. □

Now we are able to prove that the solution set of the (reformulated) truncated Stieltjes-type moment problem and the solution set of the corresponding system of the fundamental Potapov's matrix inequalities coincide.

Theorem 6.20. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Let $\mathcal{D}$ be a discrete subset of Π_+ and let $S\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Then the following statements are equivalent:*
  (i) *$S \in \mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.*
  (ii) *$P_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ and $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $z \in \Pi_+ \setminus \mathcal{D}$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Then the following statements are equivalent:*
  (iii) *$S \in \mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.*
  (iv) *$\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $z \in \Pi_+ \setminus \mathcal{D}$.*

Proof. (i)⇒(ii), (iii)⇒(iv): Use Proposition 4.9.

(ii)⇒(i): Let $m := 2n$. Observe that the function $F := \operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S$ is holomorphic. Because of (ii), the inequalities $P_{m-1}^{[F]}(z) \geq 0$ and $P_m^{[F]}(z) \geq 0$ hold true for each $z \in \Pi_+ \setminus \mathcal{D}$. From Theorem 6.5 we get then that there is a unique function $\hat{S} \in \mathcal{S}_{0,q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} \hat{S} = F$, namely $\hat{S} = S$, and that $P_k^{[\hat{S}]}(z) \geq 0$ are valid for all $k \in \mathbb{Z}_{-1,m}$ and all $z \in \mathbb{C}\setminus\mathbb{R}$. Applying Lemma 6.19, we get then (i).

(iv)⇒(iii): Let $m := 2n+1$ and use the same argumentation as in the proof of the implication "(ii)⇒(i)". □

7. Particular results on non-negative Hermitian measures

In this appendix, we summarize some facts of the integration theory of non-negative Hermitian measures. We consider a measurable space $(\Omega, \mathfrak{A})$ and use the notation $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ to denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$.

Remark 7.1. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let $f\colon \Omega \to \mathbb{C}$ be a function. Then standard arguments of measure and integration theory show that the following statements are equivalent:*
  (i) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$.*
  (ii) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, B^*\mu B; \mathbb{C})$ for all $B \in \mathbb{C}^{q\times p}$.*
  (iii) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})$ where $\tau := \operatorname{tr}\mu$ is the trace measure of μ.*

If (i) holds true, then

$$\int_A f \mathrm{d}(B^*\mu B) = B^*\left(\int_A f \mathrm{d}\mu\right)B$$

for all $A \in \mathfrak{A}$ and all $B \in \mathbb{C}^{q\times p}$.

Lemma 7.2. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let μ'_τ be a version of the Radon–Nikodym derivative of μ with respect to the trace measure $\tau := \operatorname{tr}\mu$ of μ. Let $f\colon \Omega \to \mathbb{C}$ and $g\colon \Omega \to \mathbb{C}$ be $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{C}}$-measurable functions. Then the following statements are equivalent:*

(i) $f\overline{g} \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$.

(ii) *The pair $[fI_q, gI_q]$ is left-integrable with respect to μ.*

If (i) is fulfilled, then

$$\int_\Omega f\overline{g}\mathrm{d}\mu = \int_\Omega (fI_q)\mathrm{d}\mu(gI_q)^*.$$

Lemma 7.2 can be proved by standard methods of measure and integration theory.

Remark 7.3. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let $m, n \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $p_j \in \mathbb{N}$ and let $\Phi_j\colon \Omega \to \mathbb{C}^{p_j \times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{p_j \times q}$-measurable matrix-valued function. For each $k \in \mathbb{Z}_{1,n}$, let $r_k \in \mathbb{N}$ and let $\Psi_k\colon \Omega \to \mathbb{C}^{r_k \times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r_k \times q}$-measurable matrix-valued function. Suppose that, for every choice of $j \in \mathbb{Z}_{1,m}$ and $k \in \mathbb{Z}_{1,n}$ the pair $[\Phi_j, \Psi_k]$ is left-integrable with respect to μ. Let $s, t \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $A_j \in \mathbb{C}^{s \times p_j}$, and, for each $k \in \mathbb{Z}_{1,n}$, let $B_k \in \mathbb{C}^{t \times r_k}$. Then it is readily checked that the pair*

$$[\sum_{j=1}^{m} A_j\Phi_j, \sum_{k=1}^{n} B_k\Psi_k]$$

is left-integrable with respect to μ and that

$$\int_\Omega \left(\sum_{j=1}^{m} A_j\Phi_j\right)\mathrm{d}\mu\left(\sum_{k=1}^{m} B_k\Psi_k\right)^* = \sum_{j=1}^{m}\sum_{k=1}^{n} A_j\left(\int_\Omega \Phi\mathrm{d}\mu\Psi^*\right)B_k^*.$$

Proposition 7.4. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$, let $\tau := \operatorname{tr}\mu$ be the trace measure of μ, and let μ'_τ be a version of the Radon–Nikodym derivative of μ with respect to τ. Furthermore, let $\Theta \in p \times q$-$\mathcal{L}^2(\Omega, \mathfrak{A}, \mu; \mathbb{C})$. Then:*

(a) $\mu_\Theta\colon \mathfrak{A} \to \mathbb{C}^{p \times p}$ *defined by*

$$\mu_\Theta(A) := \int_A \Theta\mathrm{d}\mu\Theta^*$$

belongs to $\mathcal{M}_{\geq}^{p}(\Omega, \mathfrak{A})$.

(b) *The non-negative Hermitian measure μ_Θ is absolutely continuous with respect to τ and $\Theta\mu'_\tau\Theta^*$ is a version of the Radon–Nikodym derivative of μ_Θ with respect to τ.*

(c) *Let $r, s \in \mathbb{N}$, let $\Phi\colon \Omega \to \mathbb{C}^{r \times p}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r \times p}$-measurable function and let $\Psi\colon \Omega \to \mathbb{C}^{s \times p}$ be an $\mathfrak{A}$-$\mathfrak{B}_{s \times p}$-measurable function. Then the pair $[\Phi, \Psi]$ is left-integrable with respect to μ_Θ if and only if the pair $[\Phi\Theta, \Psi\Theta]$ is left-integrable with respect to μ. In this case,*

$$\int_\Omega \Phi\mathrm{d}\mu_\Theta\Psi^* = \int_\Omega (\Phi\Theta)\mathrm{d}\mu(\Psi\Theta)^*.$$

Proposition 7.4 can be proved by standard arguments of measure and integration theory.

Bibliography

1. Adamyan V.M. Solution of the Stieltjes truncated matrix moment problem Solution of the Stieltjes truncated matrix moment problem / V.M. Adamyan, I.M. Tkachenko // Opuscula Math. – , 2005. – 25(1). – P. 5-24.
2. Adamyan V.M. General solution of the Stieltjes truncated matrix moment problem General solution of the Stieltjes truncated matrix moment problem. / V.M. Adamyan, I.M. Tkachenko // Operator theory and indefinite inner product spaces. – Vol. 163. – Oper. Theory Adv. Appl. – Pp. 1-22. – Birkhäuser, Basel. – 2006.
3. Albert A. Conditions for positive and nonnegative definiteness in terms of pseudoinverses / A. Albert // SIAM J. Appl. Math. – 1969. – 17. – P. 434-440.
4. Andô T. Truncated moment problems for operators Truncated moment problems for operators / T. Andô // Acta Sci. Math. (Szeged). – 1970. – 31. – P. 319-334.
5. Bolotnikov V.A. Descriptions of solutions of a degenerate moment problem on the axis and the halfaxis / V.A. Bolotnikov // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1988. – 50. – P. 25-31, i.
6. Bolotnikov V.A. Degenerate Stieltjes moment problem and associated J-inner polynomials / V.A. Bolotnikov // Z. Anal. Anwendungen. – 1995. – 14(3). – P. 441-468.
7. Bolotnikov V.A. On degenerate Hamburger moment problem and extensions of nonnegative Hankel block matrices / V.A. Bolotnikov // Integral Equations Operator Theory. – 1996. – 25(3). – P. 253-276.
8. Bolotnikov V.A. On a general moment problem on the half axis / V.A. Bolotnikov // Linear Algebra Appl. – 1997. – 255. – P. 57-112.
9. Bolotnikov V.A. On an operator approach to interpolation problems for Stieltjes functions / V.A. Bolotnikov, L.A. Sakhnovich // Integral Equations Operator Theory. – 1999. – 35(4). – P. 423-470.
10. Chen G.N. The truncated Hamburger matrix moment problems in the nondegenerate and degenerate cases, and matrix continued fractions / G.N. Chen, Y.J. Hu // Linear Algebra Appl. – 1998. – 277(1-3). – P. 199-236.
11. Chen G.N. A unified treatment for the matrix Stieltjes moment problem in both nondegenerate and degenerate cases / G.N. Chen, Y.J. Hu // J. Math. Anal. Appl. – 2001. – 254(1). – P. 23-34.
12. Chen G.N. The Nevanlinna- Pick interpolation problems and power moment problems for matrix-valued functions / G.N. Chen, X.Q. Li // Linear Algebra Appl. – 1999. – 288(1-3). – P. 123-148.
13. Choque Rivero A.E. Ein finites Matrixmomentenproblem auf einem endlichen Intervall / A.E. Choque Rivero. – Leipzig: Dissertation Dissertation, Universität Leipzig, 2001.
14. Choque Rivero A.E. A truncated matricial moment problem on a finite interval / A.E. Choque Rivero, Yu.M. Dyukarev, B. Fritzsche, B. Kirstein // Interpolation, Schur functions and moment problems. – Vol. 165. – Oper. Theory Adv. Appl. – Pp. 121-173. – Birkhäuser, Basel, 2006.
15. Dubovoj V.K. Indefinite metric in Schur's interpolation problem for analytic functions (Russian) / V.K. Dubovoj // Teor. FunktsiĭFunktsional. Anal. i Prilozhen. – I. – 1982. – 37. – P. 14-26; II. – 1982. – 38. – P. 32-39, 127; III. – 1984. – 41. – P. 55-64; IV. – 1984. – 42. – P. 46-57; V. – 1986. – 45. – P. 16-26, i; VI. – 1987. – 47. – P. 112-119.
16. Dubovoj V.K. Matricial version of the classical Schur problem, 129. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]. / V.K. Dubovoj, B. Fritzsche, B. Kirstein // B.G. Teubner Verlagsgesellschaft mbH, Stuttgart. – 1992. – With German, French and Russian summaries.
17. Dyukarev Yu.M. The Stieltjes matrix moment problem (Russian) / Yu.M. Dyukarev // Deposited in VINITI (Moscow) at 22.03.81. – No. 2628-81, 1981. – Manuscript, 37 pp.

18. Dyukarev Yu.M. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them. II (Russian) / Yu.M. Dyukarev // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1982. – 38. – P. 40-48, 127.
19. Dyukarev Yu.M. On truncated matricial Stieltjes type moment problems / Yu.M. Dyukarev, B. Fritzsche, B. Kirstein, C.M ädler // Complex Anal. Oper. Theory. – 2010. – 4(4). – P. 905-951.
20. Dyukarev Yu.M. On distinguished solutions of truncated matricial Hamburger moment problems / Yu.M. Dyukarev, B. Fritzsche, B. Kirstein, C.M ädler, H.C. Thiele // Complex Anal. Oper. Theory. – 2009. – 3(4). – P. 759-834.
21. Dyukarev Yu.M. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them. I (Russian) / Yu.M. Dyukarev, V.E. Katsnelson // Teor. FunktsiĭFunktsional. Anal. i Prilozhen. – 1981. – 36. – P. 13-27, 126.
22. Dyukarev Yu.M. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions, and interpolation problems connected with them. III (Russian) / Yu.M. Dyukarev, V.E. Katsnelson // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1984. – 41. – P. 64-70.
23. Efimov A.V. J -expanding matrix-valued functions, and their role in the analytic theory of electrical circuits (Russian) / A.V. Efimov, V.P. Potapov // Uspehi Mat. Nauk. – 1973. – 28(1(169)). – P. 65-130.
24. Elstrodt J. Maß- und I ntegrationstheorie. Springer-Lehrbuch. [Springer Textbook] / J. Elstrodt. – Berlin: Springer-Verlag, fourth, 2005. Grundwissen Mathematik. [Basic Knowledge in Mathematics].
25. Fritzsche B. On a special parametrization of matricial α-Stieltjes one-sided non-negative definite sequences. / B. Fritzsche, B. Kirstein C. Mädler // Interpolation, Schur functions and moment problems. II, 226. Oper. Theory Adv. Appl., P. 211-250. – Basel: Birkhäuser/Springer Basel AG, 2012.
26. Fritzsche B. On matrix-valued Herglotz-Nevanlinna functions with an emphasis on particular subclasses / B. Fritzsche, B. Kirstein C. Mädler // Math. Nachr. – 2012. – 285(14-15). – P. 1770-1790.
27. Fritzsche B. On a simultaneous approach to the even and odd truncated matricial Stieltjes moment problem I: A n α-Schur-Stieltjes-type algorithm for sequences of complex matrices. / B. Fritzsche, B. Kirstein C. Mädler // Linear Algebra Appl. – 2017. – 521. – P. 142-216.
28. Fritzsche B. On a simultaneous approach to the even and odd truncated matricial Stieltjes moment problem II: an α-Schur-Stieltjes-type algorithm for sequences of holomorphic matrix-valued functions / B. Fritzsche, B. Kirstein C. Mädler // Linear Algebra Appl. – 2017. – 520. – P. 335-398.
29. Fritzsche B. On matrix-valued Stieltjes functions with an emphasis on particular subclasses / B. Fritzsche, B. Kirstein C. Mädler // Large truncated T oeplitz matrices, T oeplitz operators, and related topics, 259. Oper. Theory Adv. Appl., Pp. 301-352. – Cham: Birkhäuser/Springer, 2017.
30. Fritzsche B. A Potapov-type approach to a truncated matricial Stieltjes-type power moment problem / B. Fritzsche, B. Kirstein C. Mädler, T. Makarevich. – arXiv:1712.08358 [math.CA], Dec. 2017.
31. Gesztesy F. On matrix-valued Herglotz functions On matrix-valued Herglotz functions / F. Gesztesy, E.R. Tsekanovskii // Math. Nachr. – 2000. – 218. – P. 61-138.
32. Golinski ĭ L.B. A generalization of the matrix Nevanlinna- Pick problem (Russian) / L.B. Golinski ĭ// Izv. Akad. Nauk Armyan. SSR Ser. Mat. – 1983. – 18(3). – P. 187-205.
33. Golinski ĭ L.B. On the Nevanlinna- Pick problem in the generalized Schur class of analytic matrix functions (Russian) / L.B. Golinskiĭ. – In Marchenko V.A. (ed.) // Analysis in Indefinite-Dimensional Spaces and Operator Theory, P. 23-33. – Naukova Dumka, Kiev, 1983.

34. Hu Y.J. A unified treatment for the matrix Stieltjes moment problem / Y.J. Hu, G.N. Chen // Linear Algebra Appl. – 2004. – 380. – P. 227-239.
35. Ivanchenko T.S. An operator approach to the Potapov scheme for the solution of interpolation problems / T.S. Ivanchenko, L.A. Sakhnovich // Matrix and operator valued functions. – Vol. 72. – Oper. Theory Adv. Appl. – P. 48-86. – Basel: Birkhäuser, 1994.
36. Kats I.S. On Hilbert spaces generated by monotone Hermitian matrix-functions / I.S. Kats // Har'kov Gos. Univ. Uč. Zap. 34 = Zap. Mat. Otd. Fiz.-Mat. Fak. i Har kov. Mat. Obšč. – 1951, 1950. – 4, 22. – P. 95-113.
37. Katsnelson V.E. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. I (Russian) / V.E. Katsnelson // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1981. – 36. – P. 31-48, 127.
38. Katsnelson V.E. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. II (Russian) / V.E. Katsnelson // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1982. – 37. – P. 31-48.
39. Katsnelson V.E. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. III (Russian) / V.E. Katsnelson // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1983. – 39. – P. 61-73.
40. Katsnelson V.E. Continual analogues of the Hamburger- Nevanlinna theorem, and fundamental matrix inequalities of classical problems. IV (Russian) / V.E. Katsnelson // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1983. – 40. – P. 79-90.
41. Katsnelson V.E. Methods of J theory in continuous interpolation problems of analysis. Part I. T. Ando, / V.E. Katsnelson Sapporo: Hokkaido University, 1985. – Translated from the Russian and with a foreword by T. Ando.
42. Katsnelson V.E. On transformations of Potapov's fundamental matrix inequality On transformations of Potapov's fundamental matrix inequality / V.E. Katsnelson // Topics in interpolation theory (Leipzig, 1994). – Vol. 95. – Oper. Theory Adv. Appl. – P. 253-281. – Basel: Birkhäuser, 1997.
43. Kovalishina I.V. Analytic theory of a class of interpolation problems (Russian) / I.V. Kovalishina // Izv. Akad. Nauk SSSR Ser. Mat. – 1983. – 47(3). – P. 455-497.
44. Kovalishina I.V. A multiple boundary value interpolation problem for contracting matrix functions in the unit disk (Russian) / I.V. Kovalishina // Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. – 1989. – 51. – P. 38-55.
45. Kreĭn M.G. The ideas of P. L. Č ebyšev and A. A. Markov in the theory of limiting values of integrals and their further development (Russian) / M.G. Kreĭn // Uspehi Matem. Nauk N.S. – 1951. – 6(4 (44)). – P. 3-120.
46. Kreĭn M.G. The description of all solutions of the truncated power moment problem and some problems of operator theory (Russian) / M.G. Kreĭn // Mat. Issled. – 1967. – 2(Vyp. 2). – P. 114-132.
47. Kreĭn M.G. The Markov moment problem and extremal problems. American Mathematical Society, Providence, R.I., 1977, ISBN 0-8218-4500-4. Ideas and problems of P. L. Č eby š ev and A. A. Markov and their further development, / M.G. Kreĭn, A.A. Nudel'man // Translated from the Russian by D. Louvish, Translations of Mathematical Monographs. – Vol. 50.
48. Makarevich T. Ein matrizielles M omentenproblem vom S tieljes-Typ / T. Makarevich Leipzig: Dissertation Dissertation, Universität Leipzig, 2014.
49. Rosenberg M. The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure / M. Rosenberg //Duke Math. J. – 1964, – 31. – P. 291-298.
50. Sakhnovich L.A. Interpolation theory and its applications. – Vol. 428. Mathematics and its Applications / L.A. Sakhnovich. – Dordrecht: Kluwer Academic Publishers, 1997.
51. Simon B. The classical moment problem as a self-adjoint finite difference operator / B. Simon // Adv. Math. – 1998. – 137(1). – P. 82-203.
52. Stieltjes T.J. Quelques recherches sur la théorie des quadratures dites mécaniques / T.J. Stieltjes // Ann. Sci. École Norm. Sup. – 1884. – 3, 1. – P. 409-426. – http://www.numdam.org/item?id=ASENS_1884_3_1__409_0.

53. Stieltjes T.J. Recherches sur les fractions continues Recherches sur les fractions continues / T.J. Stieltjes // Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. – 1894. – 8(4). – P. J1-J122. http://www.numdam.org/item?id=AFST_1894_1_8_4_J1_0.
54. Thiele H.C. Beiträ ge zu matriziellen P otenzmomentenproblemen / H.C. Thiele Leipzig: Dissertation Dissertation, Universität Leipzig, May, 2006.

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THE CLASSICAL ORTHOGONAL POLYNOMIALS IN RESONANT EQUATIONS

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Резюме. У статті запропоновано теорію та алгоритм для знаходження часткових розв'язків резонансних рівнянь, пов'язаних із класичними ортогональними поліномами. Це дає можливість отримати загальний розв'язок у явному вигляді. Алгоритм підходить зокрема для систем комп'ютерної алгебри, наприклад, Maple. Резонансні рівняння є невід'ємною частиною різних застосувань, наприклад, ефективного функціонально-дискретного методу (FD-метод) для розв'язування операторних рівнянь і проблеми власних значень на основі збурень і ідеї гомотопії. Ці рівняння виникають також і в контексті суперсиметричних операторів Казіміра для ді-спінової алгебри, а також рівнянь типу $A^2u = f$ з заданим оператором A в деякому банаховому просторі, наприклад, бігармонічного рівняння.

Abstract. In the present paper we propose a theory and an algorithm for particular solutions of resonant equations related to the classical orthogonal polynomials. This enable us to obtain the general solution in explicit form. The algorithm is particulary suitable for computer algebra tools like Maple. The resonant equations are an essential part of various applications e.g. of the efficient functional-discrete method (FD-method) for solving operator equations and of eigenvalue problems based on the perturbation and the homotopy ideas. These equations arose also in the context of supersymmetric Casimir operators for the di-spin algebra as well as of the equations of type $A^2u = f$ with a given operator A in some Banach space, for example, of the biharmonic equation.

1. Introduction

There are various definitions of resonant equations, see e.g. [1, 2], where a boundary value problem is called resonant, when the operator, defined by the differential equation and by the boundary conditions does not possess the inverse. In the present paper we follow the definition from [7, 16, 19] and call an equation of the form $Lf = g$ with $Lg = 0$ resonant. In other words, the right-hand side of the resonant equation belongs to the kernel $K(L)$ of the operator L. These equations are interesting both from theoretical point of view and from the practical side in various applications. For example, in [16] was proposed the so called functional-discrete method (FD-method) for solving of operator equations and of eigenvalue problems. The method is based on the

Key words. FD-method; Casimir operators; di-spin algebra; Banach space; biharmonic equation.

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ideas of perturbation of the operator involved and on the homotopy idea. This approach was applied to various problems in particulary to eigenvalue problems in [9–13] and has been proven to possess a super exponential convergence rate. An essential part of the algorithm are some inhomogeneous equations with a resonant component in the sense of the definition above. Resonant equations arise in the theory of supersymmetric Casimir operators and of di-spin algebra [7]. They can be used to study the equations of the type $A^2u = 0$ with some given operator A. Substituting $Au = v$ we reduce this equation to the pair $Av = 0, Au = v$ where the second equation is resonant.

Their importance for praxis can be explained by the following example. Let the mathematical model of some system be the operator equation

$$Au - \lambda u = f$$

in some Hilbert space H, where the system is characterized by the operator A and the parameter λ. The element f describes external perturbation. The operator A is completely defined by its eigenvalues $\lambda_1, \lambda_2, ...$ and by the corresponding eigenvectors $u_1, u_2, ...$ If the perturbation is of the kind $f = \alpha u_k$ for some fixed α, k, i.e, the equation is resonant, then the solution of the mathematical model is $u = \frac{\alpha}{\lambda_k - \lambda} u_k$. One can see that the norm $\|u\|$, which can be interpreted as "amplitude", tends to infinity as the system parameter λ tends to the so called resonant frequency λ_k. This phenomenon is called resonance and can be observed in the nature and many technical applications, e.g. in magnetic resonance imaging or nuclear spin tomography etc.

The present article deals with the resonant equations associated with the ordinary differential operators of the hypergeometric or confluent hypergeometric type, defining the classical orthogonal polynomials, i.e.

$$\mathcal{A}_n = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx} + \lambda_n \tag{1}$$

where $\sigma(x) = a_2x^2 + a_1x + a_0$ is a polynomial of the degree not greater then two, $\tau(x) = b_1x + b_0$ - a polynomial of the degree not greater then one and $\lambda_n = \lambda(n) = -nb_1 - n(n-1)a_2$ depends on the integer parameter $n \geq 0$ but not on the variable x. We consider the differential operators defining the classical orthogonal polynomials (as the first linear independent solution of the corresponding homogeneous differential equation) and the corresponding functions of the second kind (the second linear independent solution) and the resonant equations of the first and of the second kind with the corresponding right hand side. We propose a theory describing particular solutions of the inhomogeneous resonant equations. We propose a theory and an algorithm to compute such solutions, which is especially convenient for the computer algebra tools like Maple and prove that the functions generated by this algorithm satisfy the resonant differential equation. Incidently we prove a new differentiation formula which represents the derivative of a classical orthogonal polynomial through the linear combination of the same and of a neighboring polynomial and which is unified for all classical orthogonal polynomials. Its coefficients are expressed through the coefficients of $\sigma(x), \tau(x)$ and the coefficients of the recurrence relation. Such formulas are well known in the literature (see e.g. [5, 5, 24, 28]), but for each concrete orthogonal polynomial only.

2. Representation of particular solutions of resonant equations

A classical orthogonal polynomial (Jacobi, Laguerre or Hermite) $\hat{P}_n(x)$ (see e.g. [6, 24, 28]) satisfies the homogeneous differential equation

$$\mathcal{A}_n u(x) = 0 \tag{2}$$

and is called also the function of the first kind. Let $\hat{Q}_n(x)$ be the second linear independent solution of the homogeneous differential equation, which is called the function of the second kind. Then the general solution of the homogeneous differential equation (2) is given by

$$u(x) = c_1 \hat{P}_n(x) + c_2 \hat{Q}_n(x), \tag{3}$$

where c_1, c_2 are arbitrary constants.

Let us consider the resonant equations of the type

$$\mathcal{A}_n u_n(x) = R_n(x). \tag{4}$$

In the case when $R_n(x)$ is a classical orthogonal polynomial $\hat{P}_n(x)$ (the function of the first kind), the inhomogeneous differential equation (4) is called the resonant equation of the first kind. The inhomogeneous differential equation (4) with the right-hand side $\hat{Q}_n(x)$ instead of $R_n(x)$ is called the resonant differential equation of the second kind. Both functions $\hat{P}_n(x)$ and $\hat{Q}_n(x)$ satisfy the same homogeneous differential equation (2) and the same recurrence relation

$$R_{n+1}(x) = (\alpha(n)x + \beta(n))R_n(x) - \gamma(n)R_{n-1}(x), \ n = 1, 2, ... \tag{5}$$

with some coefficients $\alpha(n) = \alpha_n, \beta(n) = \beta_n, \gamma(n) = \gamma_n$ (see e.g. [6, 23, 24, 28]). If we change in the differential operator $\mathcal{A}_n$ the integer $n \geq 0$ to a real ν then the corresponding solutions $\hat{P}_\nu(x), \hat{Q}_\nu(x)$ become the hypergeometric or confluent hypergeometric functions [5,6]. Since $R_n(x)$ satisfies the homogeneous differential equation (2), then we can differentiate this equation by n in the following way: 1) switch from the integer $n \geq 0$ to a real ν, 2)differentiate by ν and 3)replace the real ν by the integer n. In regard of (1) we obtain $\mathcal{A}_n \frac{dR_n}{dn} = -\lambda'(n)R_n$ or $A_n\left(-\frac{1}{\lambda'(n)}\frac{dR_n}{dn}\right) = R_n$, which means that the function

$$u_n(x) = -\frac{1}{\lambda'(n)}\frac{dR_n}{dn} \tag{6}$$

is a particular solution of the resonant equation. Using this relation and differentiating (5) by n we obtain

$$\begin{aligned} u_{n+1}(x) = -\frac{1}{\lambda'(n+1)}\big[&-\lambda'(n)(\alpha(n)x + \beta(n))\, u_n(x) + \\ &+ \lambda'(n-1)\,\gamma(n) u_{n-1}(x) + \\ &+ \left(\alpha'(n)x + \beta'(n)\right) R_n(x) - \gamma'(n)R_{n-1}(x)\big], \ \ n = 1, 2, ... \end{aligned} \tag{7}$$

The general solution of the resonant equation (4) is given by

$$u(x) = c_1 \hat{P}_n(x) + c_2 \hat{Q}_n(x) + u_n^{(k)}(x), \tag{8}$$

where $u_n^{(k)}(x), k = 1, 2$ is a particular solution of the corresponding inhomogeneous resonant equation. Below we propose an algorithm to find the particular solutions, which is especially suitable for computer algebra tools like Maple etc. Since our algorithm below for particular solutions of the resonant differential equations of the first and of the second kind (3) is based on the same recurrence relation (5) it is valid for the resonant equations of both types and we use the notation $R_n(x)$ below for both $\hat{P}_n(x)$ and $\hat{Q}_n(x)$. The following general result on the particular solutions of the resonant equations has been proven in [19].

Theorem 1. *Let $A : X \to X$ be a linear operator acting in a Banach space X, the set $K(A) \subset X$ be the kernel of A and a connected set $\Sigma(A)$ in the complex plane be the spectral set of A. If $f(\lambda) \in K(A - \lambda E), \lambda \in \Sigma(A)$ is a differentiable function then the solution of the resonant equation*

$$(A - \lambda E)u = f(\lambda) \tag{9}$$

can be represented by

$$u(\lambda) = \frac{df(\lambda)}{d\lambda} \tag{10}$$

The proof of this theorem is based on the equivalent equation

$$(A - \lambda_0 I)\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f(\lambda)$$

with some fixed λ_0 and on passing to the limit $\lambda \to \lambda_0$.

3. An algorithm for computation of particular solutions. A general differentiation formula for classical orthogonal polynomials

Now we are at the position to formulate an algorithm for the particular solutions of the resonant equations associated with a differential operator of the hypergeometric type, defining classical orthogonal polynomials. This algorithm is especially suitable for computer algebra tools like Maple etc.

Algorithm 1. *Problem: Given a resonant equation of the first or of the second kind, return a given number N of particular solutions.*

Inputs: The number N and the right hand side $R_\nu(x)$ of the resonant equation.

Outputs: The particular solutions $u_0(x), u_1(x), ..., u_N(x)$.

1. Find

$$\chi_0(x) = -\frac{1}{\lambda'(\nu)} \left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=0}, \ \chi_1(x) = -\frac{1}{\lambda'(\nu)} \left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=1}. \tag{11}$$

Due to (6) these are particular solutions.

2. Compute $u_2(x)$ in accordance with (7) using the initial conditions

$$u_0(x) = \chi_0(x) + c_0P_0(x) + d_0Q_0(x), \ \ u_1(x) = \chi_1(x) + c_1P_1(x) + d_1Q_1(x) \tag{12}$$

with undefined coefficients c_0, c_1, d_0, d_1.

3. Find c_0, c_1, d_0, d_1 *from the condition that* $u_2(x)$ *satisfies the resonant differential equation (3).*

4. For $n = 2$ *step 1 until* $n = N$ *compute* $u_n(x)$ *by (7) and return* $u_n(x)$.

Using Theorem 1 we prove below that the sequence $u_n(x)$ generated by this algorithm satisfy the resonant equation for all $n = 0, 1, 2, ...$

Theorem 2. *All functions* $u_{n+1}(x)$ *generated by the recursion (7) with the initial conditions (12) satisfy the resonant differential equation (4).*

Proof. We use the mathematical induction and, first of all, note that the functions $u_p(x)$, $p = 0, 1, 2$ satisfy the resonant equation by construction and due to Theorem 1. Let us assume that all the functions $u_p(x), p = 0, 1, \dots, n$ satisfy the resonant differential equation (4) and prove that then the function $u_{n+1}(x)$ is its solution too.

First of all we notice that

$$\begin{aligned}
&\mathcal{A}_{n+1}u_n(x) = \sigma(x)\frac{d^2u_n}{dx^2} + \tau(x)\frac{du_n}{dx} + \lambda(n+1)u_n = \\
&= \mathcal{A}_n u_n(x) + (\lambda(n+1) - \lambda(n))u_n = R_n(x) + (\lambda(n+1) - \lambda(n))u_n, \\
&\mathcal{A}_{n+1}u_{n-1}(x) = \mathcal{A}_{n-1}u_{n-1}(x) + (\lambda(n+1) - \lambda(n-1))u_{n-1} = \\
&= R_{n-1}(x) + (\lambda(n+1) - \lambda(n-1))u_{n-1}, \\
&\frac{d}{dx}\left[(\alpha'(n)x + \beta'(n))R_n(x) - \gamma'(n)R_{n-1}(x)\right] = \\
&= \alpha'(n)R_n(x) + (\alpha'(n)x + \beta'(n))\frac{dR_n(x)}{dx} - \gamma'(n)\frac{dR_{n-1}(x)}{dx},
\end{aligned} \tag{13}$$

Further we use the differentiation formula for the classical orthogonal polynomials (which is the same for the functions of the second kind too) and which represents the derivative of these functions through the same functions of index n and the function of the index $n-1$ with some coefficients independent of x (see, e.g. [23, §4, (12)] or [6, p.171,(15); p.189, (12); p.193, (14)] for concrete classical orthogonal polynomials):

$$\begin{aligned}
\sigma(x)\frac{dR_n}{dx} &= [q_1(n)x + q_2(n)]R_n(x) + s(n)R_{n-1}(x) = \\
&= [q_{1,n}x + q_{2,n}]R_n(x) + s_n R_{n-1}(x).
\end{aligned} \tag{14}$$

Substituting this expression as well as (13)into the formula for $\mathcal{A}_{n+1}u_{n+1}(x)$, we obtain

$$\begin{aligned}
&\mathcal{A}_{n+1}u_{n+1}(x) = \frac{\lambda'(n)}{\lambda'(n+1)}(\alpha(n)x + \beta(n))(\lambda(n+1) - \lambda(n))u_n(x) \\
&\frac{1}{\lambda'(n+1)}(\alpha'(n)x + \beta'(n))(\lambda(n+1) - \lambda(n))R_n(x) + \\
&\quad + \frac{\lambda'(n)}{\lambda'(n+1)}(\alpha(n)x + \beta(n))R_n(x) -
\end{aligned} \tag{15}$$

$$
\begin{aligned}
&- \frac{2\sigma(x)}{\lambda'(n+1)} \left[-\lambda'(n)\alpha(n)\frac{du_n(x)}{dx} + \alpha'(n)\frac{dR_n(x)}{dx}\right] - \\
&- \frac{\tau(x)}{\lambda'(n+1)} \left[-\lambda'(n)\alpha(n)u_n(x) - \alpha'(n)R_n(x)\right] - \\
&- \frac{1}{\lambda'(n+1)}(\lambda'(n-1)\gamma(n)(\lambda(n+1)-\lambda(n-1))u_{n-1}(x)+ \\
&+ \lambda'(n-1)\gamma(n)R_{n-1}(x)- \\
&- \gamma'(n)(\lambda(n+1)-\lambda(n-1))R_{n-1}(x)) = \\
= &\frac{\lambda'(n)}{\lambda'(n+1)}\{\ (\alpha(n)x+\beta(n))(\lambda(n+1)-\lambda(n))+ \\
&+ 2\alpha(n)\left[q_1\left(n\right)x + q_2\left(n\right)\right] + \lambda\tau(x)\ \}\, u_n(x)+ \\
&+ \frac{\lambda'(n-1)}{\lambda'(n+1)}\{\ \gamma(n)(\lambda(n+1)-\lambda(n)) + 2\alpha(n)s\left(n\right)\ \}\, u_{n-1}(x) + \mathcal{R}(x),
\end{aligned}
$$

where $\mathcal{R}(x)$ contains the functions $R_{n-1}(x), R_n(x)$ and its derivatives but not $u_{n-1}(x), u_n(x)$. Setting the coefficients in front of $u_{n-1}(x), u_n(x)$ equal to zero, we obtain

$$
\begin{aligned}
s(n) &= -\frac{\gamma(n)}{\alpha(n)}\left[b_1 + (2n-1)a_2\right], \\
q_1(n) &= -\frac{1}{2}\left[b_1 + \lambda(n+1) - \lambda(n)\right] = na_2, \\
q_2(n) &= -\frac{b_0}{2} - \frac{\beta(n)}{2\alpha(n)}\left[\lambda(n+1)-\lambda(n)\right] = -\frac{b_0}{2} + \frac{\beta(n)}{2\alpha(n)}\left[b_1 + 2na_2\right].
\end{aligned}
\tag{16}
$$

It is easy to check that the coefficients of the differentiation formulas for all classical orthogonal polynomials satisfy (16). For example, the Laguerre polynomials are defined by the confluent hypergeometric differential equation with $\sigma(x) = a_2x^2 + a_1x + a_0 = x, \tau(x) = b_1x + b_0 = \alpha + 1 - x, \lambda_n = \lambda(n) = -nb_1 - n(n-1)a_2 = n$; i.e. $a_2 = 0, a_1 = -1, a_0 = 0, b_1 = -1, b_0 = \alpha + 1$. Besides they satisfy the recurrence relation [6, §10.2]

$$
(n+1)L_{n+1}^{\alpha}(x) - (2n+\alpha+1-x)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0, \tag{17}
$$

i.e. $\alpha(n) = -\frac{1}{n+1}, \beta(n) = \frac{2n+\alpha+1}{n+1}, \gamma(n) = -\frac{n+\alpha}{n+1}$. Due to (16) we obtain $s(n) = n+\alpha, q_1(n) = 0, q_2(n) = -\frac{\alpha+1}{2} + \frac{2n+\alpha+1}{2} = n$ and (14) implies the well known differentiation formula (see e.g. [6, §10.2])

$$
x\frac{dL_n^{\alpha}(x)}{dx} = nL_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x). \tag{18}
$$

Now, using the recurrent relation (5) we obtain from (15) the equality

$$
\mathcal{A}_{n+1}u_{n+1}(x) = R_{n+1}(x), \tag{19}
$$

which proves the assertion. □

Remark 1. *At once with (16) we have obtained the coefficients of the general differentiation formula (14) which is valid for the general classical orthogonal* *polynomials and contains all particular cases of the polynomials by Jacobi, Laguerre, Hermite known from the literature [6, 24, 28]. This formula is much more convenient for use then the corresponding formula from [23, 24].*

4. Examples

Example 1. This example demonstrates the use of Algorithm 1 for the representation of the general solution of the following Laguerre resonant equation of the first kind

$$x\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\frac{du(x)}{dx} + n\,u(x) = L_n^\alpha(x) \tag{20}$$

where

$$\begin{aligned} L_n^\alpha(x) &= \frac{(\alpha+1)_n}{n!}\Phi(-n,\alpha+1,x) = \\ &= \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\Phi(-n,\alpha+1,x) = \sum_{k=0}^{n}\begin{pmatrix} n+\alpha \\ n-k \end{pmatrix}\frac{(-x)^k}{k!} \end{aligned} \tag{21}$$

is the Laguerre polynomial satisfying the corresponding homogeneous differential equation and $\Phi(-n,\alpha+1,x)$ is the confluent hypergeometric function satisfying a degenerate form of the hypergeometric differential equation when two of the three regular singularities merge into an irregular singularity [5, p. 189, formula (14)] and $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer-Symbol.

The second linear independent solution of the homogeneous differential equation is the Laguerre function of the second kind $l_n^\alpha(x)$ (see e.g. [25, pp.16,20]). The general solution of the homogeneous Laguerre differential equation is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) \tag{22}$$

with arbitrary constants c_1, c_2. The general solution of the Laguerre resonant (inhomogeneous) equation is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x) \tag{23}$$

where c_1, c_2 are arbitrary constants and $u_n(x)$ is a particular solution of the inhomogeneous (resonant) equation.

Solving the corresponding differential equation for the Laguerre function of the second kind [25, pp.16,20] by Maple we obtain the following representation of this function for non-integer α:

$$\begin{aligned} l_n^\alpha(x) &= \Gamma(1-\alpha)L_n^\alpha(x) - (-x)^{-\alpha}\,{}_1F_1(-n-\alpha,-\alpha+1;x) = \\ &= \Gamma(1-\alpha,-x)L_n^\alpha(x) - (-x)^{-\alpha}p_n^\alpha(x)\exp(x), \\ p_{n+1}^\alpha(x) &= \frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x) - (n+\alpha)\,p_{n-1}^\alpha(x)\right], \\ &\quad n = 1,2,..., \\ p_0^\alpha(x) &= 1, p_1^\alpha(x) = 1-x. \end{aligned} \tag{24}$$

For non-negative natural $\alpha \in \mathbb{N}$ we have

$$
\begin{aligned}
&l_n^\alpha(x) = \mathrm{Ei}_1(-x)L_n^\alpha(x) - (-x)^{-\alpha}p_n^\alpha(x)\,\exp(x),\\
&p_{-1}^\alpha(x) = (\alpha-1)!,\\
&p_0^\alpha(x) = x^{\alpha-1} + x^\alpha\left[U\,(2,2,-x) + (-1)^\alpha \alpha!\,U(1+\alpha,1+\alpha,-x)\right],
\end{aligned}
\tag{25}
$$

where

$$
\mathrm{Ei}_1(x) = \int_z^\infty \frac{e^{-t}}{t}\,dt, \qquad |\mathrm{Arg}(z)| < \pi \tag{26}
$$

is the exponential integral and $U(a,b,z)$ is the Kummer's function of the second kind. The last one is a solution of the Kummer's differential equation

$$
z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0. \tag{27}
$$

The other linear independent solution of this differential equation is the Kummer's function of the first kind M defined e.g. by the hypergeometric series:

$$
M(a,b,z) = \sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!} = {}_1F_1(a;b;z). \tag{28}
$$

The Kummer's function of the second kind can be represented also as

$$
\begin{aligned}
U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}M(a,b,z)+\\
+\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a+1-b,2-b,z).
\end{aligned}
\tag{29}
$$

Note that the function at the second initial condition in (25) solves the following difference initial value problem

$$
\begin{aligned}
&p_0^\alpha(x) = x\,p_0^{\alpha-1}(x) + (\alpha-1)!,\ \ \alpha = 1,2,...,\\
&p_0^0(x) = 0.
\end{aligned}
\tag{30}
$$

Using Theorem 1 we can represent the particular solutions of the Laguerre resonant equation of the first kind also by

$$
u_n(x) = \frac{\partial}{\partial\nu}\,\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\Phi(-n,\alpha+1,x)\bigg|_{n=\nu},\ n = 0,1,... \tag{31}
$$

From this expression we extract the following particular solutions containing the elementary functions only

$$
\begin{aligned}
\chi_0^\alpha(x) = u_0(x) = -\ln(x) + \sum_{p=0}^{\alpha-1}\frac{(\alpha-p)_{p+1}}{(p+1)x^{p+1}},\\
\chi_1^\alpha(x) = u_1(x) = -L_1^\alpha(x)\ln(x) + \sum_{p=0}^{\alpha}\frac{k_p(\alpha)}{x^p},
\end{aligned}
\tag{32}
$$

where

$$k_{p+1}(\alpha) = p\sum_{i=1}^{\alpha-1} k_p(i), \;\; p = 1, 2, ..., \alpha - 1,$$
$$k_1(\alpha) = \frac{\alpha(\alpha+1)}{2}, \;\; k_0(\alpha) = -\alpha - 2, \;\; \alpha = 2, 3, ..., \tag{33}$$

At the first step of our Algorithm 1 we use the ansates

$$u_0^\alpha(x) = \chi_0^\alpha(x) + c_0 L_0^\alpha(x) + d_0 l_0^\alpha(x),$$
$$u_1^\alpha(x) = \chi_1^\alpha(x) + c_1 L_1^\alpha(x) + d_1 l_1^\alpha(x) \tag{34}$$

with undefined coefficients c_0, d_0, c_1, d_1, obtain $u_2^\alpha(x)$ from the corresponding recurrence formula of our algorithm and choose c_0, d_0, c_1, d_1 so that $u_2^\alpha(x)$ satisfies the resonant differential equation. We get $d_0 = 0, d_1 = 0$ and $c_1 = 1 + c_0$. Now one can verify that

$$u_n^\alpha(x) = -L_n^\alpha(x)\ln(x) + \frac{p_n^\alpha(x)}{x^\alpha}, \tag{35}$$

where the polynomials $p_n^\alpha(x)$ satisfy the recurrence equation

$$p_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1} p_n^\alpha(x) - \frac{n+\alpha}{n+1} p_{n-1}^\alpha(x) +$$
$$+ \frac{(\alpha-1-x)}{(n+1)^2} L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2} L_{n-1}^\alpha(x), \;\; n = 1, 2, ... \tag{36}$$

with the initial conditions

$$p_0^\alpha(x) = \sum_{p=0}^{\alpha-1} \frac{x^{\alpha-p-1}(\alpha-p)_{p+1}}{p+1} + c_0 x^\alpha,$$
$$p_1^\alpha(x) = \sum_{p=0}^{\alpha} x^{\alpha-p} k_p(\alpha) + (1+c_0) x^\alpha L_1^\alpha(x). \tag{37}$$

Example 2. Now, let us consider the Laguerre resonant equation of the second kind

$$x\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\frac{du(x)}{dx} + n\,u(x) = l_n^\alpha(x) \tag{38}$$

whith the Laguerre function of the second kind $l_n^\alpha(x)$. Due to Theorem 1 the formula

$$u_n(x) = -\frac{d}{d\nu} \;\; l_\nu^\alpha(x) \,|_{\nu=n} \tag{39}$$

defines a particular solution of (38), so that its general solution is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x). \tag{40}$$

The use of formula (39) for arbitrary n is rather burdensome, therefore we use Algorithm 1, where we for the sake of simplicity set $\alpha = 0$. Solving differential equation (38) with Maple for $n = 0$, $n = 1$ we get

$$\begin{aligned}
\chi_0(x) &= -\int_1^x \frac{\exp(t)}{t} \int_1^t \mathrm{Ei}_1(-\xi)\exp(-\xi)d\xi dt, \\
\chi_1(x) &= [(1-x)\ \mathrm{Ei}_1(-x) - \exp(x)]\times \\
&\times \int_1^x [1 + \mathrm{Ei}_1(-\xi)(-1+\xi)\exp(-\xi)](-1+\xi)d\xi + \\
&+ \int_1^x \exp(-\xi)[\mathrm{Ei}_1(-\xi)(-1+\xi) + \exp(-\xi)]^2 d\xi(-1+x).
\end{aligned} \tag{41}$$

As the ansatzes for initial values of our algorithm we use

$$u_0^0(x) = \chi_0(x) + c_0\mathrm{Ei}_1(-x) + d_0, \quad u_1^0(x) = \chi_1(x) + c_1 l_1^0(x) + d_1 L_1^0(x) \tag{42}$$

with undefined constants c_0, d_0, c_1, d_1. Differentiating the recurrence equation for the Laguerre functions of the second kind by n and in regard of (39) we obtain the following recurrence relation for particular solutions

$$\begin{aligned}
u_{n+1}^0(x) &= \frac{2n+1-x}{n+1}u_n^0(x) - \frac{n}{n+1}u_{n-1}^0(x) - \\
&- \frac{1+x}{(n+1)^2}l_n^0(x) + \frac{1}{(n+1)^2}l_{n-1}^0(x).
\end{aligned} \tag{43}$$

We substitute (42) into this equation with $n = 1$ and demand that the obtained function $u_2^0(x)$ satisfies the resonant differential equation (38) with $n = 2$, then we obtain

$$\begin{aligned}
&c_0 = -\mathrm{Ei}_1(-1)\exp(-1) - 1, \\
&d_0 = -[\mathrm{Ei}_1(-1)\exp(-1/2) + \exp(1/2)]^2, \\
&c_1 = 0, \quad d_1 = 0.
\end{aligned} \tag{44}$$

It can be verified by substitution into (43) that the following representation holds true

$$\begin{aligned}
u_n^0(x) &= p_n^0(x)\chi_1(x) + q_n^0(x)\chi_0(x) + v_n^0(x)\mathrm{Ei}_1(-x) + \\
&+ w_n^0(x)\exp(x) + q_n^0(x)d_0,
\end{aligned} \tag{45}$$

where the polynomials $p_n^0(x), q_n^0(x)$ satisfy the recurrence relation for the Laguerre polynomials with the initial conditions

$$p_0^0(x) = 0, \quad p_1^0(x) = 1, \quad q_0^0(x) = 1, \quad q_1^0(x) = 0.$$

The polynomials $w_n^0(x)$ satisfy the inhomogeneous recurrence relation for the Laguerre polynomials

$$\begin{aligned}
w_{n+1}^0(x) &= \frac{2n+1-x}{n+1}w_n^0(x) - \frac{n}{n+1}w_{n-1}^0(x) - \frac{1+x}{(n+1)^2}p_n^0(x) + \\
&+ \frac{1}{(n+1)^2}p_{n-1}^0(x), \quad n = 1, 2, ...
\end{aligned} \tag{46}$$

with the initial conditions

$$w_1^0(x) = 0, \quad w_2^0(x) = \frac{x+1}{4}.$$

The polynomials $v_n^0(x)$ solve the following discrete initial value problem

$$\begin{aligned} v_{n+1}^0(x) &= \frac{2n+1-x}{n+1}v_n^0(x) - \frac{n}{n+1}v_{n-1}^0(x) - \frac{1+x}{(n+1)^2}L_n^0(x) + \\ &+ \frac{1}{(n+1)^2}L_{n-1}^0(x), \quad n = 1,2,..., \\ & v_1^0(x) = 0, \quad v_2^0(x) = \frac{x^2-2c_0}{4}. \end{aligned} \tag{47}$$

Below we give some particular solutions of the Laguerre resonant equation of the second kind obtained by our algorithm:

$$\begin{aligned} u_0^0(x) &= \chi_0(x) + c_0\mathrm{Ei}_1(-x) + d_0,\ u_1^0(x) = \chi_1(x), \\ u_2^0(x) &= -\frac{x-3}{2}\chi_1(x) - \frac{1}{2}\chi_0(x) + \\ &+ \frac{x^2-2c_0}{4}\mathrm{Ei}_1(-x) - \frac{x^2-1}{8}\exp(x) - \frac{1}{2}d_0, \\ u_3^0(x) &= \left(\frac{1}{6}x^2 - \frac{4}{3}x + \frac{11}{6}\right)\chi_1(x) + \left(\frac{1}{6}x - \frac{5}{6}\right)\chi_0(x) + \\ &+ \left(-\frac{5}{36}x^3 + \frac{7}{12}x^2 + \frac{c_0}{6}x - \frac{5c_0}{6}\right)\mathrm{Ei}_1(-x) + \\ &+ \left(\frac{1}{24}x^3 - \frac{11}{72}x^2 - \frac{23}{72}x - \frac{1}{72}\right)\exp(x) + \left(\frac{1}{6}x - \frac{5}{6}\right)d_0, \end{aligned} \tag{48}$$

where c_0, d_0 are given by (44) and $\chi_0(x), \chi_1(x)$ – by (41).

Bibliography

1. Abdullaev A.P. On an investigation scheme of the solvability of resonant boundary value problems / A.P. Abdullaev, A.B. Burmistrova // Izvestia VUZ, Math. – 1996. – No. 11(414). – P. 14-22.
2. Abdullaev A.P. On the solvability of boundary value problems in the resonant case / A.P. Abdullaev, A.B. Burmistrova // Diff. Uravn. – 1989. – V. 25, No. 12. – P. 2044-2048.
3. Abdullaev A.P. On the generalized Green operator and the solvability of the resonant problems / A.P. Abdullaev, A.B. Burmistrova // Diff. Uravn. – 1990. – V. 26, No. 11. – P. 1860-1864.
4. Abramowitz M. Handbook of Mathematical Functions with Formulas,Grafs, and Mathematical Tables / M. Abramowitz, I. Stegun // National Bureau of Stanhdards. – Applied Mathematics Series. – 1972. – 55. – P. 1046.
5. Bateman H. Higher trancendental functions, V. 1 / H.Bateman, A. Erdélyi. – New-York, Toronto, London: McGraw-Hill Book Company, Inc., 1953.
6. Bateman H. Higher trancendental functions, V. 2 / H.Bateman, A. Erdélyi. – New-York, Toronto, London: McGraw-Hill Book Company, Inc., 1953.
7. Backhouse N.B. Resonant equations and special functions / N.B. Backhouse // Journal of Computational and Applied Mathematics. – 2001. – 133. – P. 163-169.
8. Backhouse N.B. The Resonant Legendre Equation / N.B. Backhouse // Journal of Mathematical Analysis and Applications. – 1986. – 133. – 117. – 31&317.
9. Gavrilyuk I. Super-exponentially convergent parallel algorithm for an abstract eigenvalue problem with applications to ODEs / I. Gavrilyuk, V. Makarov, N. Romaniuk // Nonl. Oscillations. – 2015. – 18, No. 3. – P. 332-356.

10. Gavrilyuk I. Superexponentially convergent algorithm for an abstract eigenvalue problem with application to ordinary differential equations / I. Gavrilyuk, V. Makarov, N. Romaniuk // J. Math Sci. – 2017. – 220. – P. 273.
11. Gavrilyuk I. Super-Exponentially Convergent Parallel Algorithm for a Fractional Eigenvalue Problem of Jacobi-Type / I. Gavrilyuk, V. Makarov, N. Romaniuk // Comput. Methods Appl. Math., ISSN (Online) 1609-9389, ISSN (Print) 1609-4840. – DOI: https://doi.org/10.1515/cmam-2017-0010.
12. Gavrilyuk I.P. Exponentially convergent parallel algorithm for nonlinear eigenvalue problems / I.P. Gavrilyuk, A.V. Klimenko, V.L. Makarov, N.O. Rossokhata // IMA J. Numer Anal. – 2007. – 27. – P. 818-838.
13. Demkiv I. Super-Exponentially convergent parallel algorithm for eigenvalue problems with fractional derivatives / I. Demkiv, I. Gavrilyuk, V. Makarov // Comput. Methods Appl. Math. – 2016. – 16(4). – P. 633-652.
14. Fedoryuk M.V. Encyclopedia of Mathematics / M.V. Fedoryuk (originator).
15. Krazer A. Transzendente Funktionen / A. Krazer, W. Franz. – Verlag: Akademie Verlag, 1960.
16. Makarov V. Hab. Thesis / V. Makarov. – Kiev: Kiev Univ.,1974.
17. Makarov V. FD-method- the exponential convergence rate / V. Makarov // Proceedings of the international conference “Informatics, numerical and applied mathematics: theory, applications, perspectives”. – Kiev: Kiev Univ., 1998.
18. Makarov V.L. On a functional-difference method of an arbitrary accuracy order for a Sturm-Liouville problem with piece-wise smooth coefficients / V.L. Makarov // DAN SSSR. – 1991. – Vol. 320, No. 1. – P. 34-39.
19. Makarov V. On the construction of partial solutions of resonant equations / V. Makarov, T. Arazmyradov // Differential Equations. – 1978. – Vol. XIV, No. 7. (in Russian).
20. Makarov V. FD-method for an eigenvalue problem in a Hilbert space with multiple eigenvalues of the base problem in a special case / V. Makarov, N. Romanyuk // Dopovidi AN of Ukraineю – 2015ю – No. 5ю – P. 26-35.
21. Makarov V. Symbolic Algorithm of the Functional-Discrete Method for a Sturm-Liouville Problem with a Polynomial Potential / V. Makarov, N. Romanyuk // Computational Methods in Applied Mathematics. – 2017. – doi:10.1515/cmam-2017-0040. arXiv:1708.03567 [math.NA]
22. Makarov V. FD-method for an eigenvalue problem with multiple eigenvalues of the base problem / V. Makarov, N. Romanyuk, I. Lazurchak // Proceedings of the Institute of Mathematics of AN of Ukraine. – 2014. – V. 11, No. 4. – P. 239-265.
23. Nikiforov F. Special Functions of the Mathematical Physics / F. Nikiforov, V. Uvarov Moscow, Nauka, 1978. (in Russian).
24. Nikiforov F. Special Functions of Mathematical Physics: A Unified Introduction with Applications / F. Nikiforov, V. Uvarov. – Basel: Springer Basel AG, 1988.
25. Parke W.C. On second solutions to second-order difference equations / W.C. Parke, L.C. Maximon. – arXiv:1601.04412 [math.CA]
26. Parke W.C. Closed-form second solution to the confluent hypergeometric difference equation in the degenerate case / W.C. Parke, L.C. Maximon // Int. J. of Difference Equations. – 2016. – Vol. 11, No. 2. – P. 203-214.
27. Rusev P.K. Hermite functions of second kind / P.K. Rusev // SERDICA. Bulgaricae mathematicae publicationes. – 1976. – Vol. 2. – P. 177-190.
28. Szegö G. Orthogonal Polynomials / G. Szegö // AMS. – 1939. – 1955.
29. Faltinsen O. Resonant sloshing in an upright annular tank / O. Faltinsen, I. Lukovsky, A. Timokha // Journal of Fluid Mechanics. – 2016. – 804. – P. 608-645. – doi: 10.1017/jfm.2016.539
30. Faltinsen O. Resonant three-dimensional nonlinear sloshing in a square-base basin. Part 4. Oblique forcing and linear viscous damping / O. Faltinsen, A. Timokha // Journal of Fluid Mechanics. – 2017. – 822. – P. 139-169. – doi: 10.1017/jfm.2017.263

31. McRobbie Donald W. MRI from Picture to Proton / Donald W. McRobbie, Elizabeth A. Moore, Martin J. Graves, Martin R. Prince. – Cambridge: Cambridge University Press. 2007.

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APPROXIMATION AND ESTIMATES IN THE PERIODIC REPRESENTATION OF REAL NUMBERS OF THE CLOSED INTERVAL $[0,5;1]$ BY A_2-CONTINUED FRACTIONS

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Резюме. В роботі знайдено оцінки наближень дійсних чисел відрізка $[0,5;1]$ ланцюговими A_2-дробами, елементи яких належать множині $\{\frac{1}{2},1\}$. Доведено, що A_2-раціональні числа (числа, що мають два різних нескінченних A_2-зображення) крім двох нескінченних, мають зліченну множину різних скінченних зображень. Спростовується гіпотеза, що кожне раціональне число є A_2-раціональним і обговорюється проблема критерія раціональності числа за його ланцюговим A_2-зображенням.

Abstract. The paper investigates the estimates of the approximations of real numbers of the closed interval $[0,5;1]$ of the A_2-continued fractions whose elements belong to set $\{\frac{1}{2},1\}$. It is proved that A_2-rational numbers (i.e. numbers that have two different infinite A_2-continued fraction representation) except two endless A_2-continued fraction representation have a countable set of different finite ones. We refute the hypothesis that every rational number is A_2-rational numbers and discuss the criterion of rationality of numbers according to its A_2-continued fraction representation.

1. Introduction

The role and importance of continued fractions in mathematics and its applications are well-known [7–9, 11, 17, 19, 20]. They are also used to develop a metric [5, 10] and probabilistic number theory [1, 6, 15, 16], the theory of dynamical systems [12], fractal geometry and fractal analysis [2, 4]. Especially well developed is the theory of elementary continued fractions whose elements are natural numbers [20]. Relatively recently, the theory of simple infinite A_2-continued fractions whose elements are positive real numbers α_0 and α_1 was created [10, 13]. It is proved that at $\alpha_0\alpha_1 = \frac{1}{2}$ the system of representation of numbers of a certain closed interval by such continued fractions, being two-character, has zero redundancy. Particular attention deserves a case when $\alpha_0 = \frac{1}{2}, \alpha_1 = 1$. We continue to develop this theory, in particular, supplement it with finite decompositions, and we focus on the interconnections of finite and infinite continued A_2-decomposers of numbers.

Let $A_2 \equiv \{\frac{1}{2},1\}$ be a two-character alphabet. Infinite continued fraction

Key words. A_2-continued fraction; A_2-rational number; criterion of rationality of number; left shift operator of digits of the A_2-continued fraction representation of number; algorithm for decomposing of rational numbers into finite A_2-continued fractions.

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} \equiv [0; a_1, a_2, \ldots, a_n, \ldots] \equiv \Delta^{A_2}_{a_1 a_2 \ldots a_n \ldots},$$

where $a_n \in A_2$, is called [5, 10] A_2-*continued fraction.*

Because $\sum\limits_{n=1}^{\infty} a_n = \infty$ then each A_2-continued fraction is convergent. Remind [20] that convergents of order n of the continued fraction $[0; a_1, a_2, \ldots, a_n, \ldots]$ is called the number $\frac{p_n}{q_n}$ which is the value of a finite continued fraction $[0; a_0; a_1, a_2, \ldots, a_n]$, that is a segment of the continued fraction moreover:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, \\ q_n = a_n q_{n-1} + q_{n-2}, \qquad n = 2, 3, \ldots; \end{cases}$$

where $p_0 = a_0, q_0 = 1, p_1 = a_1 a_0 + 1, q_1 = a_1$.

For convergents of the continued fraction the following properties are performed [20]:

1. $q_k p_{k-1} - p_k q_{k-1} = (-1)^k, \quad \forall k \in N;$
2. $\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}, \quad \forall k \in N;$
3. $q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k, \quad \forall k \in N;$
4. $\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots, a_1], \quad \forall k \in N.$

From the property (4) for A_2-continued fractions it follows that $\frac{1}{2} < \frac{q_{n-1}}{q_n} < 1$ at $n = 2, 3, \ldots$.

Theorem 1. *[10] For any $x \in [0,5; 1]$ there exists a sequence $(a_n) \in L$ such that*

$$x = [0; a_1, a_2, \ldots, a_n, \ldots], \tag{1}$$

and the numbers of a countable set can be represented as two different A_2-continued fractions:

$$x = [0; a_1, a_2, \ldots, a_n, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, a_2, \ldots, a_n, 1, (1, \frac{1}{2})], \tag{2}$$

here the round brackets mean the period.

Those numbers of the closed interval $[0, 5; 1]$ having two representation of A_2-continued fractions are called A_2-*rational numbers.* The rest of the numbers in this closed interval have only one representation and are called A_2-*irrational numbers.* The task of finding a criterion (necessary and sufficient conditions) for rationality of a number by its representation in a given coding system is traditional and for many representations is solved. Consider it for this representation.

2. Conditions of the rationality of the number by its A_2-continued fraction representation

Let denote $t = \left[0; \left(\frac{1}{2}, 1\right)\right]$ then from equality

$$t = \frac{1}{\frac{1}{2} + \frac{1}{1+t}}$$

it is easy to get equality $0{,}5t^2 + 0{,}5t - 1 = 0$ and solution of the equation $t = 1 = \left[0; \left(\frac{1}{2}, 1\right)\right]$. Similarly

$$\left[0; \left(1, \frac{1}{2}\right)\right] = \frac{1}{2}.$$

Lemma 1. *Each A_2-rational number has at least two different finite A_2-continued fraction representations that is*

$$x = [0; a_1, \ldots, a_m, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, \ldots, a_m, \frac{1}{2} + 1] = [0; a_1, \ldots, a_m, \frac{1}{2}, 1] =$$

$$= [0; a_1, \ldots, a_m, 1, (1, \frac{1}{2})] = [0; a_1, \ldots, a_m, 1 + \frac{1}{2}] = [0; a_1, \ldots, a_m, 1, 1, 1],$$

and hence it is a rational number.

Proof. Because

$$\frac{1}{2} = \left[0; \left(1, \frac{1}{2}\right)\right] \quad \text{i} \quad 1 = \left[0; \left(\frac{1}{2}, 1\right)\right],$$

then

$$[0; a_1, \ldots, a_m, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, \ldots, a_m, \frac{1}{2} + 1] =$$

$$= \frac{1}{a_1 + \ddots + \frac{1}{a_m + \frac{1}{0{,}5 + \frac{1}{1}}}} = [0; a_1, \ldots, a_m, \frac{1}{2}, 1],$$

$$[0; a_1, \ldots, a_m, 1, (1, \frac{1}{2})] = [0; a_1, \ldots, a_m, 1 + \frac{1}{2}] =$$

$$= \frac{1}{a_1 + \ddots + \frac{1}{a_m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}} = [0; a_1, \ldots, a_m, 1, 1, 1].$$

Then equalities which are indicated in the formulation of the lemma follow from the fact that equality

$$[0; a_1, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots] = [0; a_1, \ldots, a_n, a'_{n+1}, a'_{n+2}, \ldots]$$

is executed then and only then

$$[0; a_{n+1}, a_{n+2}, \ldots] = [0; a'_{n+1}, a'_{n+2}, \ldots].$$

The value of each finite A_2-continued fraction is the result of a finite number of rational actions on rational numbers. So each A_2-rational number is a rational number. □

Theorem 2. *Each A_2-rational number has a countable set of different finite A_2-continued fraction representations, in particular*

$$\frac{1}{2} = \left[0; 1, \underbrace{\frac{1}{2}, 1, \ldots, \frac{1}{2}, 1}_{2m}, 1\right], 1 = \left[0; \underbrace{\frac{1}{2}, 1, \ldots, \frac{1}{2}, 1}_{2m}, 1\right].$$

Proof. Indeed, from equality

$$1 = \frac{1}{1} = \frac{1}{\frac{1}{2} + \frac{1}{1+\frac{1}{1}}} \tag{3}$$

we have the following

$$1 = [0; 1] = [0; \frac{1}{2}, 1, 1] = [0; \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = [0; \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = \ldots$$

Then $\frac{1}{2} = [0; 1, 1] = [0; 1, \frac{1}{2}, 1, 1] = [0; 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = \ldots$.

Representations that are indicated in the lemma have the last element which is equal to 1, and hence, taking into account equality (3), we get countable set of finite representations of A_2-rational number. □

The question whether every rational number of a $[0,5;1]$ is A_2-rational is interesting. The answer to this question is directly related to another question. Is every rational number decomposed into a finite continued fraction? Let us give some examples of such expansions. But first we give the algorithm for decomposing a rational number $\frac{a}{b}$ into a A_2-continued fraction.

1. The first element a_1 of the expansion of number $x = \frac{a}{b}$ is based on the formula:

$$a_1 = \varphi\left(\frac{a}{b}\right) = \begin{cases} 1, & \text{if } \frac{1}{2} \le \frac{a}{b} \le \frac{2}{3}, \\ \frac{1}{2}, & \text{if } \frac{2}{3} \le \frac{a}{b} \le 1. \end{cases}$$

2. The following elements a_i are determined from equalities:

$$\begin{aligned} x_1 &= \frac{1}{x} - \varphi(x) = \frac{1}{x} - \frac{1}{2}\varepsilon_1 = \frac{2b - a\varepsilon_1}{2a}, \\ x_2 &= \frac{1}{x_1} - \varphi(x_1) = \frac{1}{x_1} - \frac{1}{2}\varepsilon_2, \\ &\ldots \\ x_{n+1} &= \frac{1}{x_n} - \varphi(x_n) = \frac{1}{x_n} - \frac{1}{2}\varepsilon_{n+1}, \end{aligned}$$

where $a_n = \varphi(x_{n-1}) = \frac{1}{2}\varepsilon_n, \varepsilon_n \in \{1, 2\}$.

3. The process ends if x_n becomes equal to 1 or $\frac{1}{2}$ or $\frac{1}{3}$.

- In the first case number x has $n+1$ digits, and $a_{n+1}(x) = 1$.
- In the first case number x has $n+2$ digits, and $a_{n+1}(x) = 1, a_{n+2}(x) = 1$.
- In the first case number x has $n+2$ digits, and $a_{n+1}(x) = 1, a_{n+2}(x) = \frac{1}{2}$.

The following expansions are performed:

$$\frac{2}{3} = [0;1,1,1] = \left[0;\frac{1}{2},1\right], \frac{3}{4} = \left[0;1,1,\frac{1}{2}\right], \frac{4}{5} = \left[0;\frac{1}{2},1,1,\frac{1}{2}\right],$$

$$\frac{5}{6} = \left[0;\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,1,1,1,\frac{1}{2}\right].$$

$$\frac{6}{7} = \left[0;\frac{1}{2},1,1,1\right] = \left[0;\frac{1}{2},\frac{1}{2},1\right],$$

$$\frac{7}{8} = \left[0;\frac{1}{2},1,1,\frac{1}{2},1,1,\frac{1}{2},\right], \ \frac{8}{9} = \left[0;\frac{1}{2},1,1,1,1,1\right],$$

$$\frac{9}{10} = \left[0;\frac{1}{2},1,1,1,1,1,\frac{1}{2}\right],$$

$$\frac{5}{6} = \left[0;\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,1,\frac{1}{2}\right)\right].$$

Notation. The number $x = \frac{5}{6}$ has both finite and periodic A_2-expansion and the later does not satisfy the definition of a A_2-rational number. This fact refutes the hypothesis that every rational number is A_2-rational number. It remains neither proven nor disproved the hypothesis that every rational number has a finite A_2-continued fraction expansion.

3. Left shift operator on digits of A_2-continued fraction representation of number

In the space of A_2-continued fraction representations we define operator ω by equality

$$\omega(\Delta^{A_2}_{a_1a_2\ldots}) = \Delta^{A_2}_{a_2a_3\ldots}, \tag{4}$$

called *left shift operator on digits of A_2-continued fraction representation of number.*

Let us use only the first of the two existing representations (2) of A_2-rational number. Then from equality (4) we get well-defined function of number $x = [0;a_1,a_2,\ldots]$ that has the following analytical form

$$\omega(x) = \frac{1}{x} - a_1(x) = \frac{1-a_1x}{x}.$$

Let

$$\omega^n(x) = \underbrace{\omega(\omega(\ldots\omega}(x))) = \frac{u_nx+v_n}{c_nx+d_n},$$

then

$$\omega^n(x) = \frac{1}{\omega^{n-1}(x)} - a_n(x) = \frac{1-\omega^{n-1}a_n(x)}{\omega^{n-1}}.$$

Then $u_0 = 1, \quad v_0 = 0, \quad c_0 = 0, \quad d_0 = 1$ and at $a_n = \frac{1}{2}$ we have

$$\frac{u_{n+1}x+v_{n+1}}{c_{n+1}x+d_{n+1}} = \frac{c_nx+d_n}{u_nx+v_n} - \frac{1}{2} = \frac{(2c_n-u_n)x+2d_n-v_n}{2u_nx+2v_n},$$

hence the following

$$\begin{cases} u_{n+1} = 2c_n - u_n, \\ v_{n+1} = 2d_n - v_n, \\ c_{n+1} = 2u_n, \\ d_{n+1} = 2v_n. \end{cases}$$

If $a_n = 1$ then

$$\frac{u_{n+1}x + v_{n+1}}{c_{n+1}x + d_{n+1}} = \frac{c_n x + d_n}{u_n x + v_n} - 1 = \frac{(c_n - u_n)x + d_n - v_n}{u_n x + v_n}.$$

We have

$$\begin{cases} u_{n+1} = c_n - u_n, \\ v_{n+1} = d_n - v_n, \\ c_{n+1} = u_n, \\ d_{n+1} = v_n. \end{cases}$$

Let's estimate the value of $|u_n|$ above.

Theorem 3. *The following inequality is being performed:*

$$c_n \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^n - \left(\frac{1-\sqrt{17}}{4} \right)^n \right), \quad \forall n \in Z_+.$$

Proof. It is clear that $(u_0; v_0; c_0; d_0) = (1; 0; 0; 1)$. Possible options for $(u_1; v_1; c_1; d_1)$ are $(-1; 2; 2; 0)$ and $(-1; 1; 1; 0)$ for $a_1 = \frac{1}{2}$ or 1 respectively. The following cases are possible:

$$u_{n+1} = 2c_n - u_n = \begin{cases} 2u_{n-1} - u_n, \\ 4u_{n-1} - u_n, \end{cases}$$

and

$$u_{n+1} = c_n - u_n = \begin{cases} 2u_{n-1} - u_n, \\ u_{n-1} - u_n. \end{cases}$$

So

$$u_{n+1} = ku_{n-1} - u_n,$$

where $k \in \{1; 2; 4\}$.

We got

$$|u_{n+1}| = |ku_{n-1} - u_n| \leq |ku_{n-1}| + |u_n| \leq 4|u_{n-1}| + |u_n|.$$

Let (s_n) such a sequence that

$$s_{n+1} = s_n + 4s_{n-1}, \quad \forall n \in N,$$

$$s_0 = 1, \quad s_1 = 1.$$

It is inductively easy to show that

$$u_n \leq s_n, \quad \forall n \in Z_+.$$

Because

$$s_n = \frac{1}{\sqrt{17}}\left(\left(\frac{1+\sqrt{17}}{4}\right)^n - \left(\frac{1-\sqrt{17}}{4}\right)^n\right),$$

then

$$u_n \le \frac{1}{\sqrt{17}}\left(\left(\frac{1+\sqrt{17}}{4}\right)^n - \left(\frac{1-\sqrt{17}}{4}\right)^n\right), \quad \forall n \in Z_+.$$

It is clear that $c_n = ku_n$, where $k \in \{\ 1; \frac{1}{2}\}$, then

$$c_n \le u_n \le \frac{1}{\sqrt{17}}\left(\left(\frac{1+\sqrt{17}}{4}\right)^n - \left(\frac{1-\sqrt{17}}{4}\right)^n\right), \quad \forall n \in Z_+. \qquad \square$$

4. Properties of numbers with periodic A_2-continued fractions representation

Theorem 4. *If number y has a period in its A_2-continued fractions representation then it looks $y = \alpha + \sqrt{\gamma}$ where $\alpha, \gamma \in \mathbb{Q}$.*

Proof. Let $y = [0, \alpha_1, \alpha_2, \ldots, \alpha_k, (\beta_1, \ldots \beta_l)]$, then we have

$$[0, (\beta_1, \ldots, \beta_l)] = \omega^k(y) = \frac{u_k y + v_k}{c_k y + d_k},$$

$$[0, (\beta_1, \ldots, \beta_l)] = \omega^l\left(\frac{u_k y + v_k}{c_k y + d_k}\right).$$

So,

$$\frac{u_{k+l} y + v_{k+l}}{c_{k+l} y + d_{k+l}} = \frac{u_k y + v_k}{c_k y + d_k},$$

hence the following

$$y^2(c_{k+l}u_k - u_{k+l}c_k) + y(u_k c_{k+l} + v_k d_{k+l} - u_{k+l}d_k - v_{k+l}c_k) + v_k d_{k+l} - v_{k+l}d_k = 0,$$

which proves necessary. $\square$

Theorem 5. *If equation $ax^2 + bx + c = 0$, $\quad (a, b, c \in \mathbb{Z}, \quad a \neq 0)$ has a solution $x_1 = \alpha + \sqrt{\gamma}$, where $\alpha, \gamma \in \mathbb{Q}$, $\quad \sqrt{\gamma} \notin \mathbb{Q}$, then it has a solution $x_2 = \alpha - \sqrt{\gamma}$.*

Proof. It is clear that

$$a(\alpha^2 + 2\alpha\sqrt{\gamma} + \gamma) + b(\alpha + \sqrt{\gamma}) + c = 0,$$

$$\sqrt{\gamma}(2a\alpha + b) + a\alpha^2 + a\gamma + c + b\alpha = 0.$$

If $2a\alpha + b \neq 0$, then $\sqrt{\gamma} \in \mathbb{Q}$ and we get a contradiction.

So,

$$\begin{cases} 2a\alpha + b = 0, \\ a\alpha^2 + a\gamma + c + b\alpha = 0. \end{cases}$$

Hence we get

$$\begin{aligned} ax_2^2 + bx_2 + c &= a(\alpha^2 - 2\alpha\sqrt{\gamma} + \gamma) + b(\alpha - \sqrt{\gamma}) + c = \\ &= -\sqrt{\gamma}(2\alpha a + b) + a\alpha^2 + a\gamma + c + b\alpha = 0. \end{aligned} \qquad \square$$

Theorem 6. *If number*

$$y = \frac{e}{f} + \sqrt{\frac{g}{h}} \quad \in [0{,}5; 1],$$

where $l, f, g, h \in \mathbb{N}$, $(g; h) = (f; h) = 1$, $\sqrt{\frac{g}{h}} \notin \mathbb{Q}$, *has* A_2*-continued fractions representation of the form*

$$y = [0, (\beta_1, \ldots, \beta_l)],$$

then the following inequality is being performed:

$$h \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^l - \left(\frac{1-\sqrt{17}}{4} \right)^l \right).$$

Proof. We have

$$\frac{u_l y + v_l}{c_l y + d_l} = y$$

hence

$$c_l y^2 + (d_l - u_l) y - v_l = 0.$$

By the theorem 5 the last equation also has a root

$$\tilde{y} = \frac{e}{f} - \sqrt{\frac{g}{h}},$$

then

$$-\frac{b_l}{c_l} = y\tilde{y} = \frac{e^2}{f^2} - \frac{g}{h} = \frac{l^2 h - g f^2}{f^2 h}$$

hence

$$-b_l f^2 h = c_l (l^2 h - g f^2).$$

The left side of the last equality is divided by h hence c_l is divided by h.

Taking into account theorem 3, we have

$$h \leq |c_l| \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^l - \left(\frac{1-\sqrt{17}}{4} \right)^l \right). \qquad \square$$

5. Approximation of real number of the closed interval $[0,5;1]$ by A_2-continued fractions

Let $\nu_1(x,n) = \frac{l_n}{n}, \nu_{\frac{1}{2}}(x,n) = \frac{k_n}{n}$, where l_n i k_n is the number of elements 1 i $\frac{1}{2}$ respectively among $(a_1, \ldots, a_n)$ in A_2-continued fraction representation of number $x = [0; a_1, \ldots, a_n, \ldots]$.

Let's call the values $\lim\limits_{n\to\infty} \nu_{\frac{1}{2}}(x,n) = \nu_{\frac{1}{2}}$ and $\lim\limits_{n\to\infty} \nu_1(x,n) = \nu_1$ by the frequencies of the digits 1 and $\frac{1}{2}$ in A_2-continued fraction representation of x, provided that these limits exist.

Lemma 2. *Let B be a set of sets of numbers $(\alpha_1, \ldots, \alpha_{k+l})$ among which l elements are equal to 1 and k elements are equal to $\frac{1}{2}$, and let $q((\alpha_1, \ldots, \alpha_{k+l}))$ be a number that is defined by the following recurrence formula:*

$$q_0 = 1, \quad q_1 = \alpha_1, \quad q_n = \alpha_n q_{n-1} + q_{n-2}, \quad n = 2, 3, \ldots, k+l,$$

$$q_{k+l} = q((\alpha_1, \ldots, \alpha_{k+l})).$$

Then there exist such constants $D_j, \tilde{D}_j (j \in \{1,2,3,4\})(D_j, \tilde{D}_j > 0)$, that do not depend k and l such that

$$\min_{(\beta_1,\ldots,\beta_{k+l})} = D_1\delta_1^l\eta_1^k + D_2\delta_1^l\eta_2^k + D_3\delta_2^l\eta_1^k + D_4\delta_2^l\eta_2^k. \tag{5}$$

$$\max_{(\beta_1,\ldots,\beta_{k+l})} = \tilde{D}_1\delta_1^l\eta_1^k + \tilde{D}_2\delta_1^l\eta_2^k + \tilde{D}_3\delta_2^l\eta_1^k + \tilde{D}_4\delta_2^l\eta_2^k. \tag{6}$$

where $\delta_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}, \eta_{1,2} = \dfrac{1 \pm \sqrt{17}}{4}$.

Proof. Let $q_k = \frac{1}{2}q_{k-1} + q_{k-2}, q_{k+1} = q_k + q_{k-1}$, then

$$q_{k+1} = \frac{1}{2}q_{k-1} + q_{k-2} + q_{k-1} = 1,5q_{k-1} + q_{k-2}.$$

If $q_k = q_{k-1} + q_{k-2}, q_{k+1} = \frac{1}{2}q_k + q_{k-1}$, then

$$q_{k+1} = \frac{1}{2}q_{k-1} + \frac{1}{2}q_{k-2} + q_{k-1} = 1,5q_{k-1} + 0,5q_{k-2}.$$

As we see, in the first case, the value of q_{k+1} is greater than in the second case.

Let $c_n(\beta_0, \beta_1), d_n(\gamma_0, \gamma_1)$ be such sequences that

$$c_{n+1}(\beta_0, \beta_1) = c_n(\beta_0, \beta_1) + c_{n-1}(\beta_0, \beta_1), \quad \forall n \in N,$$
$$c_0(\beta_0, \beta_1) = \beta_0, c_1(\beta_0, \beta_1) = \beta_1.$$
$$d_{n+1}(\gamma_0, \gamma_1) = \frac{1}{2}d_n(\gamma_0, \gamma_1) + d_{n-1}(\gamma_0, \gamma_1), \quad \forall n \in N,$$
$$d_0(\gamma_0, \gamma_1) = \gamma_0, d_1(\gamma_0, \gamma_1) = \gamma_1.$$

Inductively on n it is easy to show that

$$c_n(\tilde{\beta_0}, \tilde{\beta_1}) > c_n(\beta_0, \beta_1),$$
$$d_n(\tilde{\gamma_0}, \tilde{\gamma_1}) > d_n(\gamma_0, \gamma_1), \quad \forall n \in N,$$

if $\tilde{\beta_j} > \beta_j > 0,\ \tilde{\gamma_j} > \gamma_j > 0,\ \forall j \in \{0; 1\}$.

Considering all the above, we obtain that when replacing the neighboring elements $(\frac{1}{2}, 1)$ on $(1, \frac{1}{2})$ in a set $(\alpha_1, \ldots, \alpha_{k+l})$ we will reduce the value of the expression $q(\alpha_1, \ldots, \alpha_{k+l})$. We will make such a replacement as long as possible. As a result, we will come to the set

$$\left(\underbrace{1, 1, \ldots, 1}_{l}, \underbrace{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}}_{k}\right).$$

So,

$$\min_{(\alpha_1,\ldots,\alpha_{k+l})\in B} q((\alpha_1,\ldots,\alpha_{k+l})) = q\left((\underbrace{1,1,\ldots,1}_{l},\underbrace{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}_{k})\right).$$

Easy to see that

$$d_n(\gamma_0;\gamma_1) = \frac{4\gamma_1+(\sqrt{17}-1)\gamma_0}{2\sqrt{17}}\left(\frac{1+\sqrt{17}}{4}\right)^n + \\ + \frac{(\sqrt{17}+1)\gamma_0-4\gamma_1}{2\sqrt{17}}\left(\frac{1-\sqrt{17}}{4}\right)^n,$$

$$c_n(\beta_0;\beta_1) = \frac{2\beta_1+(\sqrt{5}-1)\beta_0}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \\ + \frac{(\sqrt{5}+1)\beta_0-2\beta_1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad \forall\in N.$$

It is clear that $q\left(\left(\underbrace{1,1,\ldots,1}_{l},\underbrace{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}_{k}\right)\right)$ is determined by the system of equations

$$q_0=1, q_1=1, q_2=q_1+q_0, q_3=q_2+q_1,\ldots,q_l=q_{l-1}+q_{l-2},$$

$$q_{l+1}=\frac{1}{2}q_l+q_{l-1}, q_{l+2}=\frac{1}{2}q_{l+1}+q_l,\ldots,q_{k+l}=\frac{1}{2}q_{k+l-1}+q_{k+l-2}.$$

Then we have

$$q_l=\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^l+\frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^l.$$

$$q_{l-1}=\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{l-1}+\frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{l-1}.$$

$$q_{k+l}=\frac{4b_1^*+(\sqrt{17}-1)b_0^*}{2\sqrt{17}}\left(\frac{1+\sqrt{17}}{4}\right)^k+\frac{(1+\sqrt{17})b_0^*-4b_1^*}{2\sqrt{17}}\left(\frac{1-\sqrt{17}}{4}\right)^k,$$

where $b_1^*=q_l, b_0^*=q_{l-1}$.

Similarly, from a system of equations

$$q_0=1, q_1=\frac{1}{2}, q_2=\frac{1}{2}q_1+q_0,\ldots,q_k=\frac{1}{2}q_{k-1}+q_{k-2},$$

$$q_{k+1}=q_k+q_{k-1},\ldots,q_{k+l}=q_{k+l-1}+q_{k+l-2}$$

we have

$$b_0^*=E_1\eta_1^k+E_1\eta_2^k, \qquad b_1^*=\tilde{E}_1\eta_1^k+\tilde{E}_2\eta_2^k,$$

$$\max_{(\beta_1,\dots,\beta_{k+l})\in B} q((\beta_1,\dots,\beta_{k+l})) =$$
$$= \frac{2b_1^* + (\sqrt{5}-1)b_0^*}{2\sqrt{5}}\delta_1^l + \frac{(\sqrt{5}+1)b_0^* - 2b_1^*}{2\sqrt{5}}\delta_2^l$$

for some constants $E_1, E_2, \tilde{E}_1, \tilde{E}_2$, which are easily determined, and from here we have (6). □

Lemma 3. *If number $x = [0; a_1, a_2, \dots, a_n, \dots]$ is A_2-rational number, then*

$$\overline{\lim_{n\to\infty}} \frac{q_n}{\delta_1^{l_n}\eta_1^{k_n}} \le \tilde{D}_1,$$
$$\underline{\lim_{n\to\infty}} \frac{q_n}{\delta_1^{l_n}\eta_1^{k_n}} \ge D_1.$$

Proof. It is clear that $\lim\limits_{n\to\infty} l_n = +\infty$, because otherwise number x will be A_2-rational number. The same is true for the (k_n). We get

$$\lim_{n\to\infty} \frac{\delta_1^{l_n}\eta_2^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}} = \lim_{n\to\infty}\left(\frac{\eta_2}{\eta_1}\right)^{k_n} = 0,$$
$$\lim_{n\to\infty} \frac{\delta_2^{l_n}\eta_1^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}} = \lim_{n\to\infty}\left(\frac{\delta_2}{\delta_1}\right)^{l_n} = 0,$$
$$\lim_{n\to\infty} \left|\frac{\delta_2^{l_n}\eta_2^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}}\right| \le \lim_{n\to\infty}\frac{|\eta_2|^n}{\eta_1^n} = \lim_{n\to\infty}\left(\frac{|\eta_2|}{\eta_1}\right)^n = 0.$$

Taking into account lemma 2 we get what we need. □

Theorem 7. *If for A_2-continued fraction representation of irrational number x frequencies of digits $\frac{1}{2}$ and 1 exist, which are equal відповідно $\nu_{\frac{1}{2}}$ and ν_1 respectively then for any $\varepsilon > 0$ there is a number n_0 such that*

$$|x - \frac{p_n}{q_n}| < \frac{1}{(\delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}} - \varepsilon)^{2n+1}}, \quad \forall n \ge n_0,$$

in particular for any irrational number $y \in [0,5;1]$ there exist a number n_1 and constant C such that

$$|y - \frac{p_n}{q_n}| < \frac{C}{(\frac{1+\sqrt{17}}{4})^{2n+1}}, \quad \forall n \ge n_1.$$

Proof. Taking into account lemma 2 we get $\sqrt[n]{q_n} \to \delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}} (n \to +\infty)$. Then for any for any sufficiently small $\varepsilon > 0$ we get $q_n > (\delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}} - \varepsilon)^n$ starting with a certain number n_0.

Given the inequality

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

we have the required inequality.

Let us consider the function $g(x) = \left(\frac{1+\sqrt{5}}{2}\right)^x \left(\frac{1+\sqrt{17}}{4}\right)^{1-x}$ interval $[0;1]$. It is obvious that the function $g(x)$ continuous on $[0;1]$. Since the function

$\ln(g(x)) = x\ln\left(\frac{1+\sqrt{5}}{2}\right) + (1-x)\ln\left(\frac{1+\sqrt{17}}{4}\right)$ is increasing $\left(\frac{1+\sqrt{5}}{2} > \frac{1+\sqrt{17}}{4}\right)$ then $g(x)$ is increasing too.

Taking into account lemma 2 we get that

$$q_n \geq D_1(\delta_1^{\nu_1 n}\eta_1^{\frac{\nu_1}{2}n})^n \geq D_1(g(0))^n = D_1\left(\frac{1+\sqrt{17}}{4}\right)^n,$$

starting with a certain number n_1. This implies that we need. □

Bibliography

1. Albeverio S. On singularity and fine spectral structure of random continued fractions / S. Albeverio, Y. Kulyba, M. Pratsiovytyi, G. Torbin // Math. Narch. – Vol. 288, Issue 16. – P. 1803-1813.
2. Cusick T.W. Hausdorf dimension of sets of continued fractions / T.W. Cusick // Quan. J. Math. Oxford. – 1990. – 2, 41. – P. 277-286.
3. Hensley D. Continued fraction Cantor sets, Hausdorff Dimension and Functional Analysis / D. Hensley // Journal of number theory. – 1992. – 40. – P. 336-358.
4. Hirst K.E. Fractional dimension theory of continued fractions / K.E. Hirst // Quart. J. Math. – 1970. – 21. – P. 29-35.
5. Pratsiovytyi M. Properties of the distribution of the random variable defined by A_2-continued fraction with independent elements / M. Pratsiovytyi, D. Kyurchev // Random Operators and Stochastic Equations. – 2009. – Vol. 17, No. 1. – P. 91–101.
6. Albeverio S. On singularity and spectral structure of distributions of random continued fractions / S. Albeverio, Y. Kulyba, M. Pratsiovytyi, G. Torbin // Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Drahomanova. – Ser. 1. – Fiz.-Mat. Nauky [Trans. Nats. Pedagog. Mykhailo Drahomanov Univ. Ser. 1 Phys. Math]. – 2010. – N.13(1). – P. 16-31. (in Ukrainian).
7. Arnold V.I. Continued fractions / V.I. Arnold. – Moscow: MCCME, 2001. – 40 p. (in Russian).
8. Bodnar D.I. Branched continued fractions / D.I. Bodnar. – Kyiv: Nauka, 1986. – 176 p. (in Russian).
9. Bodnarchuk P.I. Branched continued fractions and their applications / P.I. Bodnarchuk, V.Ya. Skorobagatko. – Kyiv: Naukova Dumka, 1974. – 272 p. (in Ukrainian).
10. Dmitrenko S.O. A_2-continued fraction representation of real number and it's geometry / S.O. Dmitrenko, D.V. Kyurchev, M.V. Pratsiovytyi // Ukrainian Math. J. – 2009. – Vol. 61, № 4. – P. 452-463. (in Ukrainian).
11. Jones W.B. Continued Fractions. Analytic Theory and Applications / W.B. Jones, W.J. Thron. – Moscow: Mir, 1985. – 414 p. (in Russian).
12. Kac M. Statistical independence in probability, analysis, and number theory / M. Kac. – Moscow: Foreign Literature Publishing House, 1963. – 156 p.
13. Pratsiovytyi M.V. Singularity of the distribution of a random variable given by an A_2-continued fraction with independent elements / M.V. Pratsiovytyi, D.V. Kyurchev // Theory of Probability and Mathematical Statistics. – 2009. – №81. – P. 139-154.
14. Leshchinsky O.L. Cantor projectors on continued fractions / O.L. Leshchinsky // Fractal analysis and related issues. – 1998. – № 1. – P. 76-83. (in Ukrainian).
15. Leshchinsky O.L. One class of singular distributions of random variables represented by an elementary continued fraction with independent elements / O.L. Leshchinsky, M.V. Pratsiovytyi // Modern physical and mathematical researches of young scientists of Ukrainian universities. – Kyiv: Kyiv Nats. T. Shevchenko Univ. – 1995. – P. 20-30. (in Ukrainian).
16. Pratsiovytyi M.V. Singularity of distributions of random variables given by distributions of its continued fraction representation / M.V. Pratsiovytyi // Ukrainian Math. J. – 1996. – 48, № 8. – P. 1086-1095. (in Ukrainian).

17. Pratsiovytyi M.V. Fractal approach to the study of singular distributions / M.V. Pratsiovytyiю – Kyiv: Nats. Pedagog. Mykhailo Drahomanov Univ., 1998. – 296 p. (in Ukrainian).
18. Skorobagatko V.Ya. The theory of branched continued fractions and its application in computational mathematics / V.Ya. Skorobagatkoю – Moscow: Nauka, 1983. – 312 p. (in Russian).
19. Savyavko M.S. Integral continued fractions / M.S. Savyavkoю – Kyiv: Naukova dumka, 1994. – 205 p. (in Ukrainian).
20. Khinchin A.Ya. Continued fractions / A.Ya. Khinchinю – Moscow: Nauka, 1978. – 116 p. (in Russian).

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REPLACEMENTS IN THE FINITE ELEMENT METHOD FOR THE PROBLEM OF ADVECTION-DIFFUSION-REACTION

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Резюме. У даній роботі запропоновано новий підхід до числового розв'язування сингулярно-збурених задач адвекції-дифузії-реакції (АДР). Цей підхід базується на експоненціальних прямій і зворотній замінах до і після варіаційного формулювання, відповідно. Одержано теоретичні результати існування розв'язку та порядку збіжності. Проведено числові експерименти для сингулярно-збурених задач АДР. Наведено графіки одержаних розв'язків у стаціонарному та нестаціонарному випадках, таблиці похибок та експериментальний порядок збіжності запропонованого методу.

Abstract. In this work, a new approach for the numerical approximation of the solution for the initial-boundary problem of advection-diffusion-reaction (ADR) is proposed. This approach is based on exponential direct and inverse replacements, before and after variation formulations, respectively. Theoretical results of the existence of the solution and of the order of convergence are obtained. Numerical experiments are conducted for singularly perturbed ADR problems. Graphs of the obtained results for stationary and non-stationary problems, table of errors and experimental orders of convergence are presented.

1. Introduction

The mathematical modeling of processes of advection-diffusion-reaction (ADR) is the relevant area of research. However, in the case of large advantage of advection coefficients over diffusion coefficients, the standard approach based on the finite element method (FEM) leads to the loss of stability of the approximation. Nowadays, many approaches to solving singularly perturbed ADR problems might be found in works of M. Ainsworth, N. Bahvalov, I. Babuska, G. Marchuk, Ya. Savula, G. Shynkarenko, S. Wang and others. In particular, among the approaches well known are an application of the exponential basis and exponential weights [6], [9], functions bubbles basis [5] in the FEM. Among the well-known approaches, there are also adaptive schemes of FEM [1], [10].

The problem of improving the stability of FEM to solve the problem of ADR, despite a large number of publications, is still opened. Among a large number of existing methods, there is a question of choosing the optimal method for improving sustainability. This fact may be the subject of another review publication. The authors propose a new approach to solving this actual problem,

Key words. Advection-diffusion-reaction; finite element method; exponential replacement.

which does not require the use of irregular grids, h-p adaptive grids, counter-flow schemes, etc., which might greatly complicate the programming of the method.

Let there Ω is a bounded limited area in R^2 with a Lipschitz boundary Γ. The problem is to find c — an unknown concentration, which satisfies a differential equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (Vc) - \nabla \cdot (K \cdot \nabla c) + \sigma c = f(x,t); \quad x \in \Omega, \quad t \in (0,T] \tag{1}$$

an initial condition

$$c(x,0) = 0; \quad x \in \bar{\Omega} \tag{2}$$

and a boundary condition

$$\nu \cdot (K \cdot \nabla c) + \lambda c = \psi; \quad x \in \Gamma, \quad t \in (0,T]. \tag{3}$$

In (1),(3) $V = (V_1, V_2)$ is a velocity vector of constant values $V_1 > 0$, $V_2 > 0$, K is a diffusivity coefficient, σ is a coefficient of reaction, λ is a constant value, f is a function of external sources, ψ is a function defined on the boundary Γ and $\nu = (l_1, l_2)$ is a directed vector to Γ. Coefficients are positive, constant and dimensionless and, because V_1, V_2 are constant, environment is incompressible $\nabla \cdot (V) = 0$.

An operator of the problem was considered

$$Ac = \nabla \cdot (V \cdot c) - \nabla \cdot (K \cdot \nabla c) + \sigma c.$$

Therefore, the following equation has been considered

$$\frac{\partial c}{\partial t} + Ac = f$$

with initial and boundary conditions (2), (3), respectively.

2. FEM WITH EXPONENTIAL REPLACEMENT

Previously, using a numerical experiment, it was found that the solution obtained by the standard FEM with linear and quadratic basis functions [1,5–10] is unstable in the case of a singular perturbed problem. In this paper, a new alternative approach to solving the singular perturbed ADR problems is proposed.

In (1)-(3) the following replacement [4] was applied

$$c = u \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right). \tag{4}$$

Therefore, the problem (1)-(3) will be equivalent to the following problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - K\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2}\right) + \left(\frac{V_1^2 + V_2^2}{4K} + \sigma\right) u = \\ &= f \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right), \quad x \in \Omega; \end{aligned} \tag{5}$$

$$K\frac{\partial u}{\partial \nu} + \left(\left(\frac{V_1}{2} l_1 + \frac{V_2}{2} l_2\right) + \lambda\right) u = \psi \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right), \quad x \in \Gamma; \tag{6}$$

$$u(x,0)=0; \quad x\in\bar{\Omega}.$$

The next step is a variation formulation of the resulting problem. To do this, space $W = W_2^{(1)}(\Omega)$ was introduced. Then, equation (5) was multiplied on arbitrary function $w \in W$ and integrated over the area Ω

$$\begin{gathered}\int\limits_{\Omega}\frac{\partial u}{\partial t}wd\Omega - K\int\limits_{\Omega}\Delta uwd\Omega + \left(\frac{V_1^2+V_2^2}{4K}+\sigma\right)\int\limits_{\Omega}uwd\Omega = \\ = \int\limits_{\Omega} fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered} \tag{7}$$

To the first term of the equation (7) the Green's formula for Laplacian [2] was applied. Thus, the following expression was obtained

$$\begin{gathered}\int\limits_{\Omega}\frac{\partial u}{\partial t}wd\Omega + K\int\limits_{\Omega}\nabla u\nabla wd\Omega - K\int\limits_{\Gamma}\frac{\partial u}{\partial \nu}wd\Gamma + \\ + \left(\frac{V_1^2+V_2^2}{4K}+\sigma\right)\int\limits_{\Omega}uwd\Omega = \int\limits_{\Omega} fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered} \tag{8}$$

According to the algorithm, the discretization of the problem based on the division of the area Ω by finite elements and then on the construction of approximations using a linear combination of basic functions might be the next step. However, after direct applying of the discretization, the initial system of linear algebraic equations (SLAE) will have different orders of the coefficients of right and left parts. That is due to the last integrant multiplier on the right side of (8). Therefore, an approximation of the solution might be unstable.

That is the main reason why a reverse replacement was proposed to be applied in (8)

$$u = c\,\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right). \tag{9}$$

Then, because

$$\frac{\partial u}{\partial x_i} = \frac{\partial c}{\partial x_i}\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) - \frac{V_i}{2K}c\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)$$

the following expression was obtained

$$\begin{gathered}K\int\limits_{\Omega}\nabla u\nabla wd\Omega = K\int\limits_{\Omega}\nabla c\nabla w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega - \\ - \sum_{i=1,2}\frac{V_i}{2}\int\limits_{\Omega} c\frac{\partial w}{\partial x_i}\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered} \tag{10}$$

The formula is known [2]

$$\int\limits_{\Omega}\left(\frac{\partial \varphi}{\partial x_1}+\frac{\partial \psi}{\partial x_2}\right)d\Omega = \int\limits_{\Gamma}(\varphi l_1 + \psi l_2)\,d\Gamma,$$

then, taking: $\varphi = uv, \psi = 0$ and vice versa, it is easy to make sure that

$$\int\limits_{\Omega} v\frac{\partial u}{\partial x_i}d\Omega = -\int\limits_{\Omega} u\frac{\partial v}{\partial x_i}d\Omega + \int\limits_{\Gamma} uvl_i d\Gamma.$$

Therefore, the following transformation was applied to the last two terms in expression (10)

$$\begin{aligned}
&-\frac{V_i}{2}\int\limits_{\Omega} c\frac{\partial w}{\partial x_i}\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega = \\
&= \frac{V_i}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_i}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega - \\
&-\int\limits_{\Gamma} cwl_i\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) - \\
&-\frac{V_i^2}{4K}\int\limits_{\Omega} cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.
\end{aligned} \tag{11}$$

According to the boundary condition (6)

$$\begin{aligned}
-K\int\limits_{\Gamma}\frac{\partial u}{\partial \nu}wd\Gamma &= \int\limits_{\Gamma}\left(\frac{V_1}{2}l_1+\frac{V_2}{2}l_2\right)uwd\Gamma + \int\limits_{\Gamma}\lambda uwd\Gamma - \\
&-\int\limits_{\Gamma}\psi\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)wd\Gamma.
\end{aligned}$$

Further, taking into account the inverse replacement (9), the following expression was obtained

$$\begin{aligned}
-K\int\limits_{\Gamma}\frac{\partial u}{\partial \nu}wd\Gamma &= \int\limits_{\Gamma}\left(\frac{V_1}{2}l_1+\frac{V_2}{2}l_2\right)cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma + \\
+\int\limits_{\Gamma}\lambda cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma &- \int\limits_{\Gamma}\psi\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)wd\Gamma.
\end{aligned} \tag{12}$$

Finally, after combining expressions (7) - (12), the variation formulation of problem was obtained. To find such $c(x,t) \in L_2(0,T;W)$ that satisfies the following equation $\forall w \in W$

$$\int\limits_{\Omega}\frac{\partial c}{\partial t}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega + K\int\limits_{\Omega}\nabla c\nabla w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega +$$

$$+\frac{V_1}{2}\int\limits_{\Omega}\frac{\partial c}{\partial x_1}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega +$$

$$+\frac{V_2}{2}\int\limits_{\Omega}\frac{\partial c}{\partial x_2}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$
$$+\int\limits_{\Gamma}\lambda cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma+\sigma\int\limits_{\Omega}cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega= \tag{13}$$
$$=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+\int\limits_{\Gamma}\psi w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma.$$

It is important to notify that variation formulation (13) is significantly different from the formulation obtained by using the classical approach for obtaining variation formulation. Coefficients V_1 and V_2 at advection integral expressions are divided by 2. Integral expressions on the left and the right sides have the same order.

According to the procedure of FEM, the triangulation of the area Ω by finite elements $\Omega\approx\bigcup\limits_{i=0}^{N}\Omega_i$ with boundary elements $\Gamma\approx\bigcup\limits_{i=1}^{M}\Gamma_i$ was obtained. Then, on the each finite element Ω_e with vertices numbering i,j,k an approximation of the solution was built by using linear basic functions [8]:

$$c_h=c_i^h\varphi_i^{(e)}(x_1,x_2)+c_j^h\varphi_j^{(e)}(x_1,x_2)+c_m^h\varphi_m^{(e)}(x_1,x_2), \tag{14}$$

where $\varphi_i^{(e)}(x_1^{(i)},x_2^{(i)})=\frac{1}{\delta}\left(a_i+b_ix_1+c_ix_2\right)$ and $a_i=x_1^{(j)}x_2^{(m)}-x_1^{(m)}x_2^{(j)}$, $b_i=x_2^{(j)}-x_2^{(m)}$, $c_i=x_1^{(m)}-x_1^{(j)}$, $\delta=2S_{ijm}$.

Then the following bilinear forms were introduced

$$m(c',w)=\int\limits_{\Omega}\frac{\partial c}{\partial t}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega;$$

$$a(c,w)=K\int\limits_{\Omega}\nabla c\nabla w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$
$$+\sum_{i=1,2}\frac{V_i}{2}\int\limits_{\Omega}\frac{\partial c}{\partial x_i}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$
$$+\int\limits_{\Gamma}\lambda cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma+\sigma\int\limits_{\Omega}cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega;$$

$$l(w)=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+\int\limits_{\Gamma}\psi w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma.$$

Therefore, by application semi-discrete Galerkin's method with

$$c_h(x,t)=\sum_{j=1}^{N}c_j\left(t\right)\varphi_j^h\left(x\right)$$

the following Cauchy problem was formulated

$$\begin{cases} \sum\limits_{j=1}^{N} \left\{ m_{ij} C_j'(t) + a_{ij} C_j(t) \right\} = l_i(t), \ t \in (0,T], \ i = \overline{1,N}; \\ \sum\limits_{j=1}^{N} m_{ij} C_j(0) = p_i, \ i = \overline{1,N} \end{cases} \tag{15}$$

where $m_{ij} = m\left(\varphi_i^h, \varphi_j^h\right)$; $a_{ij} = a\left(\varphi_i^h, \varphi_j^h\right)$; $l_i(t) = l\left(\varphi_i^h\right)$; $p_i = m\left(c_0, \varphi_i^h\right)$.

To discretize the problem (15) by time variable the Euler's method [8] was applied. Mesh partitioning step δ was introduced. Thus, the following recurrence scheme was obtained

$$\begin{gathered} \sum_{j=1}^{N} \{m_{ij} C_j(t_{k+1})\} = \sum_{j=1}^{N} \{m_{ij} C_j(t_k)\} + \\ +\delta \left\{ l_i(t_k) - \sum_{j=1}^{N} a_{ij} C_j(t_k) \right\}, \ i = \overline{1,N}; \\ \sum_{j=1}^{N} m_{ij} C_j(t_0) = p_i, \ i = \overline{1,N}; \end{gathered} \tag{16}$$

where $k = \overline{1,N_t}$, N_t is a number of subintervals by time variable.

It should be noted that, according to the specifics of the proposed approach, FEM ultimately leads to solving the SLAE with the specific coefficients. These coefficients are the sum of integrals, which will include exponential function. It is known that for such integrals using classic quadrature in practice gives a high error of the approximation. Therefore, we propose to use special IOST quadrature [3], which is an extended Gaussian quadrature. The proposed in [3] formula completely avoids the crowding of Gaussian points and allows to obtain approximate values of the integrals determined with the high accuracy. The last is shown in [3] for exponential integrant functions.

3. Convergence analysis and error estimate

For the purpose of theoretical study, a stationary problem with homogeneous Dirichlet boundary conditions was considered

$$\nabla \cdot (Vc) - \nabla \cdot (K \cdot \nabla c) + \sigma c = f(x); \quad x \in \Omega,$$

$$c = 0, \quad x \in \Gamma.$$

3.1. **Classical approach FEM (linear basis).** According to the classical approach, the following variation formulation was obtained: find $c \in W$ that

$$\begin{aligned} K \int\limits_{\Omega} \nabla c \nabla w d\Omega + V_1 \int\limits_{\Omega} \frac{\partial c}{\partial x_1} w d\Omega + V_2 \int\limits_{\Omega} \frac{\partial c}{\partial x_2} w d\Omega + \\ + \sigma \int\limits_{\Omega} c w d\Omega = \int\limits_{\Omega} f w d\Omega, \quad \forall w \in W. \end{aligned} \tag{17}$$

Bilinear form was defined

$$\tilde{a}(c,w) = K\int\limits_{\Omega} \nabla c \nabla w d\Omega + V_1 \int\limits_{\Omega} \frac{\partial c}{\partial x_1} w d\Omega + V_2 \int\limits_{\Omega} \frac{\partial c}{\partial x_2} w d\Omega + \sigma \int\limits_{\Omega} c w d\Omega.$$

Theorem 1. *The bilinear form $\tilde{a}(c,w)$ is continuous, i.e.* $\exists M > 0:$

$$\tilde{a}(c,w) \le M\|c\|_{W_2^{(1)}}\|w\|_{W_2^{(1)}}.$$

$M = \max\left\{\sqrt{3}K, \sqrt{3}\max\{V_1,V_2\}, \sqrt{3}\sigma, 1\right\}.$

Proof. Norm in Sobolev's space is $\|u\|^2_{W_2^{(1)}} = \int\limits_{\Omega}\left(u^2 + (\nabla u)^2\right) d\Omega$. Expression for $(\tilde{a}(c,w))^2$ was considered and evaluated by using elementary inequality $(q-p)^2 \ge 0 \quad \Rightarrow \quad 2qp \le q^2 + p^2$.

$$(\tilde{a}(c,w))^2 = \left(\int\limits_{\Omega}\left(K\nabla c \nabla w + \sum_i V_i \frac{\partial c}{\partial x_i} w + \sigma c w\right)\right)^2 \le$$

$$\le \int\limits_{\Omega}\left(3(K\nabla c\nabla w)^2 + 3(\max\{V_1,V_2\}\nabla c w)^2 + 3(\sigma c w)^2\right) d\Omega.$$

Let's reinforce inequality by adding an integral term

$$\int\limits_{\Omega} c^2 (\nabla w)^2 d\Omega \ge 0$$

$$(\tilde{a}(c,w))^2 \le \int\limits_{\Omega}\Big(3(K\nabla c\nabla w)^2 + 3(\max\{V_1,V_2\}\nabla c w)^2 +$$

$$+3(\sigma c w)^2 + (c\nabla w)^2\Big) d\Omega \le\le M^2\|c\|^2_{W_2^{(1)}}\|w\|^2_{W_2^{(1)}}. \qquad \square$$

Obviously, in the case $V_1 >> K$ and(or) $V_2 >> K$, $M = \sqrt{3}\max\{V_1,V_2\}$.

Theorem 2. *The bilinear form $\tilde{a}(c,w)$ is V-elliptic, i.e.* $\exists m > 0 : \tilde{a}(c,c) \ge m\|c\|^2_{W_2^{(1)}}$.

$$m = \min\{K,\sigma\}.$$

Proof. It is known [8] that a bilinear form $b(c,w) = \int\limits_{\Omega}\left(V_1 \frac{\partial c}{\partial x_1} w + V_2 \frac{\partial c}{\partial x_2} w\right) d\Omega$ is skew-symmetric, i.e. $b(c,w) = -b(w,c)$. Therefore, $b(c,c) = 0$. Then

$$(\tilde{a}(c,c)) = \int\limits_{\Omega}\left(K\nabla c\nabla c + \sum_i V_i \frac{\partial c}{\partial x_i} c + \sigma c^2\right) =$$

$$= \int\limits_{\Omega}\left(K(\nabla c)^2 + \sigma c^2\right) \ge m\|c\|^2_{W_2^{(1)}}. \qquad \square$$

Thus, the following two-sided estimate of bilinear form was obtained

$$m\|c\|^2_{W_2^{(1)}} \le \tilde{a}(c,c) \le M\|c\|^2_{W_2^{(1)}}.$$

Consequences If the function $f(x) \in L_2(\Omega)$, then, according to the Lax-Milgram's theorem [8], there is a single weak solution of the variation problem (17). In addition, by using Cea's lemma and theorem about the order of convergence proved in [8], applying the FEM with linear basis functions (14), a priori estimation of the error of approximate solution c_h to an exact solution c was obtained

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{M}{m} \|c\|_{W_2^{(2)}}.$$

3.2. **Method of exponential replacements.** According to the approach proposed in this paper, taking into account the homogeneous boundary condition

$$\begin{aligned} a(c,w) = K \int\limits_{\Omega} \nabla c \nabla w \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega + \\ + \sum_i \frac{V_i}{2} \int\limits_{\Omega} \frac{\partial c}{\partial x_i} w \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega + \\ + \sigma \int\limits_{\Omega} cw \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega. \end{aligned} \tag{18}$$

Theorem 3. *The bilinear form $a(c,w)$ is continuous, i.e. $\exists Q > 0:$*

$$a(c,w) \leq Q \|c\|_{W_2^{(1)}} \|w\|_{W_2^{(1)}}.$$

Proof. An expression for $(a(c,w))^2$ was considered and Cauchy-Schwarz's inequality was applied

$$\begin{aligned} (a(c,w))^2 = \\ = \left(\int\limits_{\Omega} \left(K \nabla c \nabla w + \sum_i \frac{V_i}{2} \frac{\partial c}{\partial x_i} w + \sigma c w \right) \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega \right)^2 \leq \\ \leq \int\limits_{\Omega} \left(K \nabla c \nabla w + \sum_i \frac{V_i}{2} \frac{\partial c}{\partial x_i} w + \sigma c w \right)^2 d\Omega \int\limits_{\Omega} \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right)^2 d\Omega. \end{aligned} \tag{19}$$

Let's evaluate the last multiplier

$$\begin{aligned} \int\limits_{\Omega} \exp\left(-\frac{V_1x_1 + V_2x_2}{K}\right) d\Omega \leq \left\{ \max_{\Omega} \exp\left(-\frac{V_1x_1 + V_2x_2}{K}\right) \right\} S_\Omega = \\ = \left\{ \min_{\Omega} \exp\left(\frac{V_1x_1 + V_2x_2}{K}\right) \right\} S_\Omega, \end{aligned}$$

where S_Ω is a square of the area Ω. Let's evaluate the first multiplier of the right side of (19) by introducing notation $L = \frac{1}{2}\max\{V_1, V_2\}$ and using elementary inequality $2qp \le q^2 + p^2$.

$$\int\limits_\Omega \left(K\nabla c\nabla w + \sum_i \frac{V_i}{2}\frac{\partial c}{\partial x_i} w + \sigma c w \right)^2 d\Omega \le$$
$$\le \int\limits_\Omega (K\nabla c\nabla w + L\nabla c w + \sigma c w)^2 d\Omega \le$$
$$\le \int\limits_\Omega \left(3(K\nabla c\nabla w)^2 + 3(L\nabla c w)^2 + 3(\sigma c w)^2 \right) d\Omega.$$

Let's reinforce inequality by adding an integral term

$$\int\limits_\Omega c^2 (\nabla w)^2 d\Omega \ge 0$$

$$\int\limits_\Omega \left(K\nabla c\nabla w + \sum_i \frac{V_i}{2}\frac{\partial c}{\partial x_i} w + \sigma c w \right)^2 d\Omega \le$$
$$\le \int\limits_\Omega \left(3(K\nabla c\nabla w)^2 + 3(L\nabla c w)^2 + (c\nabla w)^2 + 3(\sigma c w)^2 \right) d\Omega \le$$
$$\le 3K^2 \int\limits_\Omega (\nabla c)^2 d\Omega \int\limits_\Omega (\nabla w)^2 d\Omega + 3L^2 \int\limits_\Omega (\nabla c)^2 d\Omega \int\limits_\Omega (w)^2 d\Omega +$$
$$+ \int\limits_\Omega (c)^2 d\Omega \int\limits_\Omega (\nabla w)^2 d\Omega + 3\sigma^2 \int\limits_\Omega (c)^2 d\Omega \int\limits_\Omega (w)^2 d\Omega \le$$
$$\le P^2 \int\limits_\Omega \left(c^2 + (\nabla c)^2 \right) d\Omega \int\limits_\Omega \left(w^2 + (\nabla w)^2 \right) d\Omega,$$

$P = \max\left\{ \sqrt{3}K, \sqrt{3}L, \sqrt{3}\sigma, 1 \right\}$. Obviously, in the case of the singularly perturbed problem $P = \frac{\sqrt{3}}{2}\max\{V_1, V_2\}$.

Therefore, the following evaluation was obtained

$$(a(c,w))^2 \le Q^2 \|c\|^2_{W_2^{(1)}} \|w\|^2_{W_2^{(1)}}$$

and

$$Q = \sqrt{\left\{ \min_\Omega \exp\left(\frac{V_1 x_1 + V_2 x_2}{K} \right) \right\} S_\Omega P}. \qquad \square$$

Theorem 4. *The bilinear form $a(c,w)$ is V-elliptic, i.e. $\exists q > 0 : a(c,c) \ge q\|c\|^2_{W_2^{(1)}}$.*

Proof. Let's investigate the bilinear form

$$b(c,w) = \frac{V_1}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_1} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega + \\ + \frac{V_2}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_2} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Taking into account the homogeneous boundary conditions

$$\sum_i \frac{V_i}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_i} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega = \\ = \sum_i \frac{V_i}{2}\int\limits_{\Omega} \frac{\partial w}{\partial x_i} c \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega + \\ + \sum_i \frac{V_i^2}{4K}\int\limits_{\Omega} cw \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Then

$$b(c,w) = -b(w,c) + \sum_i \frac{V_i^2}{4K}\int\limits_{\Omega} cw \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Therefore,

$$b(c,c) = \sum_i \frac{V_i^2}{8K}\int\limits_{\Omega} c^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega$$

and

$$a(c,c) = K\int\limits_{\Omega} (\nabla c)^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega + \\ + \left(\frac{V_1^2+V_2^2}{8K}+\sigma\right)\int\limits_{\Omega} c^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega \geq \\ \geq \mu \int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega,$$

$\mu = \min\left\{K, \left(\frac{V_1^2+V_2^2}{8K}+\sigma\right)\right\}$. Obviously, in the case of the singularly perturbed problem $\mu = K$.

$$\int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega \geq \\ \geq \min_{\Omega} \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) \int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) d\Omega =$$

$$= \max_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right) \int_{\Omega} \left((\nabla c)^2 + c^2\right) d\Omega.$$

Therefore,

$$q = \mu \max_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right). \qquad \square$$

Consequences If the function $f(x) \in L_2(\Omega)$, then, according to the Lax-Milgram theorem [8], there is a single weak solution of the variation problem (19). In addition, by using Cea's lemma and theorem about the order of convergence proved in [8], applying the FEM with linear basis functions (14), a priori estimation of the error of the approximate solution c_h to the exact solution c was obtained

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{Q}{q} \|c\|_{W_2^{(2)}}.$$

In the case that $V_1 >> K$ and(or) $V_2 >> K$ **classical approach of FEM** gives an error

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3} \max\{V_1, V_2\}}{\min\{K, \sigma\}} \|c\|_{W_2^{(2)}}. \tag{20}$$

And **method of exponential replacements** gives an error

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3}}{2} \frac{\max\{V_1, V_2\} \sqrt{\left\{\min\limits_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{K}\right)\right\} S_\Omega}}{K \max\limits_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right)} \|c\|_{W_2^{(2)}}.$$

Considering that the region Ω is in the first quarter of the coordinate system

$$\left\{\min_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{K}\right)\right\} = 1.$$

Therefore,

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3}}{2} \frac{\max\{V_1, V_2\} S_\Omega}{K \max\limits_{\Omega} \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right)} \|c\|_{W_2^{(2)}}. \tag{21}$$

On the right sight of evaluation (20), a maximum of advection coefficients appears, which in the case of singularly perturbed problems might be a high number. This is the main reason for the loss of stability by using the classical FEM approach. On the other hand, in the evaluation (21) the value in the denominator of the corresponding constant value is much higher than in the numerator and balances this issue.

Thus, the order of the convergence is preserved in both methods, but the constant at h in the method of exponential replacements is much smaller. Therefore, at the same value of step, an estimate of the error of the proposed method is much better than without replacements.

4. Numerical Results

Numerical experiments were conducted for different ADR problems. In this paper stationary and non-stationary cases were considered.

4.1. **Sationary problem.** For the purpose of study of experimental order of convergence, a stationary one-dimensional problem on $[0,1]$ with homogeneous Dirichlet boundary conditions was considered. In this case, the exact solution is known

$$c(x) = \frac{f}{\sigma}\left\{\left(\frac{e^{\alpha_2 b}-1}{e^{\alpha_1 b}-e^{\alpha_2 b}}\right)e^{\alpha_1 x} + \left(\frac{1-e^{\alpha_1 b}}{e^{\alpha_1 b}-e^{\alpha_2 b}}\right)e^{\alpha_2 x} + 1\right\}, \qquad (22)$$

$$\alpha_{1,2} = \frac{-V \pm \sqrt{V^2+4K\sigma}}{-2K}.$$

The relative error of the method was calculated by the following formula

$$R_h = \max_i \frac{|c(x[i]) - c_h(x[i])|}{c(x[i])} * 100\%.$$

In the Table 1 we show relative errors with different advection coefficients and numbers of mesh points. For the rest of input parameters the following values were set $K = 1.0$; $\sigma = 1.0$; $f = 1.0$. As can be seen from Table 1 relative

TABL. 1. Relative errors

N	$V = 70$	$V = 100$	$V = 150$
16	0.045904065	0.033463525	0.012854694
32	0.035734439	0.044157740	0.042586510
64	0.018337116	0.026518133	0.037617746
128	0.014365357	0.017101069	0.022781162

error of the exact and approximate solution is extremely small and decreases with an increase in the number of mesh points.

To calculate the experimental order of the convergence, the following scheme was applied. Approximations c_{h_1}, c_{h_2} were calculated on 2 grids for $h_1, h_2 = 0.5h_1$, respectively.

Denotation $e_i = \|c - c_{h_i}\|$, $i = 1,2$ was introduced. Then, orders of convergence in the output spaces $W_2^{(1)}(\Omega)$ and $L_2(\Omega)$ were calculated according to the formula

$$p \approx \frac{\ln e_1 - \ln e_2}{\ln h_1 - \ln h_2}.$$

Corresponding orders of convergence are not presented for $N = 20$, $V = 1$ and $N = 80$, $V = 100$ because results on 2 grids are needed to calculate the orders. From the results obtained, the experimental order of convergence coincides with the theoretical one obtained in the preceding paragraph of the article.

4.2. **Non-stationary problem.** The same area and boundary conditions as in the previous example were considered. The scheme (16) was applied. On the (Fig. 1) an exact solution and approximations of the solution of problem (1)-(3) in different moments of time are presented. The number of mesh points

TABL. 2. Orders of convergence

V	N	$\|c_h - c\|_{W_2^{(1)}}$	$\|c_h - c\|_{L_2}$	order p in $W_2^{(1)}$	order p in L_2
1	10	$0,02764197$	$0,00081619$	–	–
	20	$0,01375605$	$0,00020139$	$1,0067934$	$2,0189291$
	40	$0,00681524$	$5,16259 \cdot 10^{-5}$	$1,0132307$	$1,9638124$
100	20	$0,0547536$	$0,0007214$	–	–
	40	$0,0398390$	$0,0002793$	$0,4587726$	$1,3689093$
	80	$0,0231478$	$8,26024 \cdot 10^{-5}$	$0,7833058$	$1,7576934$
	160	$0,0116643$	$1,95351 \cdot 10^{-5}$	$0,9887709$	$2,0801149$
	320	$0,0053174$	$3,88396 \cdot 10^{-6}$	$1,1332922$	$2,330465$

$N = 128$,mesh partitioning step by time variable $\delta = 0.05$. Input parameters were set into the following values

$$V = 100; \quad K = 1.0; \quad \sigma = 1.0; \quad f = 1 - e^{-t}.$$

It is obviously that solution coincides with an exact solution (22) at $t \to \infty$. Graphs 1, 2, 3 are approximated concentrations c_h in moments of time $t = 0.1, t = 0.2, t = 0.3$, respectively; Graphs 4, 5, 6 are approximated concentrations c_h in moments $t = 0.8, t = 1.0, t = 2$, respectively; Graphs 7, 8, 9 are approximated concentrations c_h in moments $t = 3, t = 4.5, t = 5$, respectively; Graph 10 is an exact solution (22) of the problem (1)-(3) at $t \to \infty$.

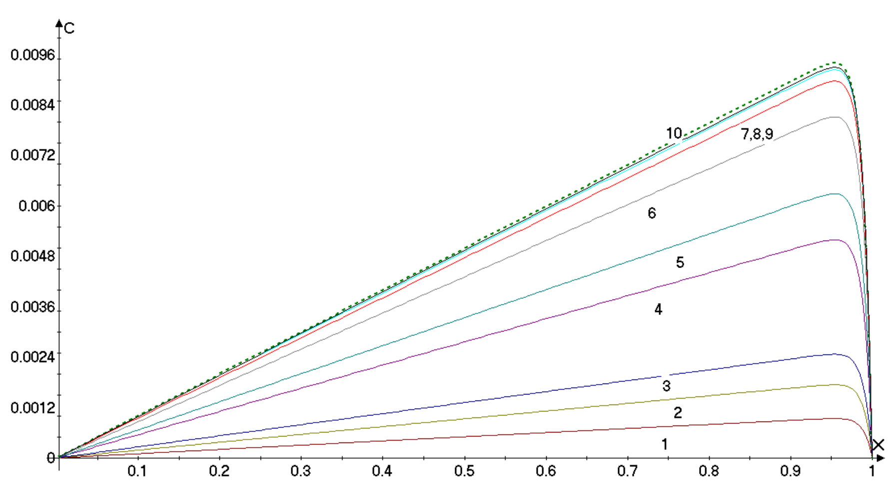

FIG. 1. Approximations in different moments of time and an exact solution

As can be seen from (Fig. 1), approximations of the unknown solution exactly coincide with the solution of a stationary problem with increasing moments of time.

The concentration closer to the end of interval $[0, 1]$ in the fixed point $x = 0.875$ is shown on the (Fig. 2). This is a point where, in fact, there is a problem in the case of significant advection coefficients, overcome by the method proposed in this paper. Coefficients of diffusion, reaction, right part f and the number of mesh points are the same as in the previous example.

On the graph 1 coefficient of advection $V = 70$, on graph 2 coefficient of advection $V = 100$, on graph 3 coefficient of advection $V = 150$.

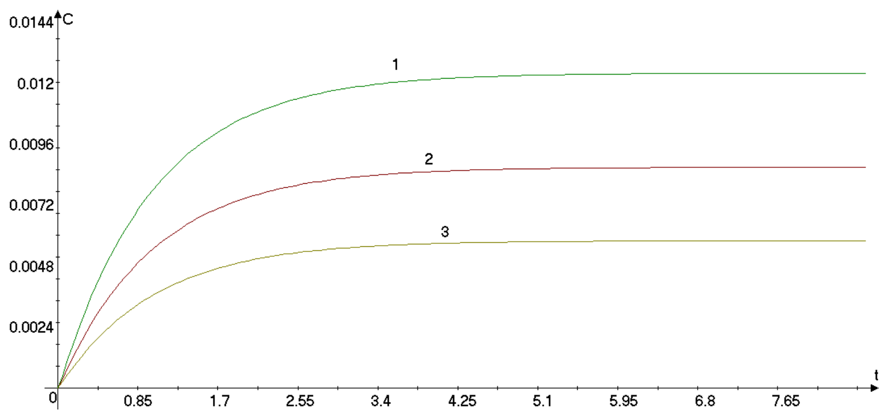

FIG. 2. Approximations in the fixed point $x = 0.875$

As can be seen from obtained results, the solution coincides with the solution of the stationary problem, that is, the process becomes stationary. It is also worth noting that the value of the desired concentration c at the fixed point $x = 0.875$ decreases with an increase in the advection coefficient, which corresponds to the nature of the phenomenon, as well as the fact that with an increase of V, obtained approximation reaches stationary behavior faster.

5. Conclusions

Thus, in this paper, a singular perturbed initial-boundary problem of ADR has been considered. A new alternative method based on exponential direct and reverse replacement in FEM for resolving singular-perturbed problems of ADR has been proposed.

The sequence of theorems have been proved and the existence of the solution and order of convergence of the proposed method have been shown.

Numerical experiments have been conducted and results have been compared with an exact solution, known in partial case. Obtained results have proved the effectiveness of the proposed method.

In the long term, it is planned to apply the proposed method to the mathematical models of the distribution of drugs and others in which the aforementioned specificity of the coefficients arises.

Bibliography

1. Babuska I. The adaptive finite element method / I. Babuska. – Austin: TICAM Forum Notes no 7– University of Texas, 1997.
2. Fichtenholz G. Course of differential and integral calculus / G. Fichtenholz. – Moscow: Science, 1966. – Vol. 3.
3. Hussain F. Accurate Evaluation Schemes for Triangular Domain Integrals / F. Hussain, M.S. Karim // IOSR Journal of Mechanical and Civil Engineering. – 2012. – Vol. 2. – P. 38-51.
4. Kartashov E. Analytical methods in heat conduction theory / E. Kartashov. – Moscow: High school, 1985.
5. Kukharskyy V. Modified method of residual-free bubbles for solving the advection-diffusion problem with hight Peclet number / V. Kukharskyy, Y. Savula, I. Kryven // Visnyk of the Lviv University. Series Appl. Math. and Informatics. – 2013. – Vol. 20. – P. 85-94. (in Ukrainian).
6. Mandzak T. Mathematical modeling and numerical analysis of the advection-diffusion in heterogeneous environment / T. Mandzak, Y. Savula. – Lviv: Spline, 2009. (in Ukrainian).
7. Turchyn Y. Computer modelling of the advection-diffusion of drugs in the living tissues / N. Kit, Ya. Savula, Y. Turchyn // Visnyk of the Lviv University. Series Appl. Math. and Informatics. – 2013. – Vol. 19. – P. 93-98. (in Ukrainian).
8. Savula Ya. Numerical analysis of problems of mathematical physics by variation methods / Ya. Savula. – Lviv: I. Franko National University, 2004. (in Ukrainian).
9. Sinchuk Y. Exponential approximations of FEM for the singular-perturbed problems of convection-diffusion-reaction / Y. Sinchuk, G. Shynkarenko // Visnyk of the Lviv University. Series Appl. Math. and Informatics. – 2007. – Vol. 12. – P. 157-169. (in Ukrainian).
10. Sinchuk Y. Adaptive schemes of finite element method for the singular perturbed variation problems of convection-diffusion / Y. Sinchuk // Physic-mathematical modelling and informational technologies. –2008. – Vol. 7. – P. 95-102. (in Ukrainian).

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ON ACOUSTIC EQUILIBRIA

E. V. Tkachenko, A. N. Timokha

Резюме. Стаття узагальнює математичну теорію віброрівноваги на випадок акустично-керованої поверхні розділу між випаровним газом та рідиною в контейнері.

Abstract. The present paper generalises mathematical theory of vibroequilibria onto the case of the acoustically-driven interface between ullage gas and liquid in a container.

1. Introduction

Using high-frequency vibrations and acoustic waves for the contactless control of a limited liquid volume is a relatively-old technologic idea coming from the 70-90's. In this context, one should mention the so-called acoustical levitation (of liquid drops) utilised in chemical and pharmaceutical industries as well as for getting ultra-pure (smart) materials [4, 6, 15]. A mathematical theory of acoustically-levitated liquid drops can be found in [5]. Other popular studies deal with mean (time-averaged) shapes of the contained liquid in tanks undergoing a high-frequency vibration. These are associated with novel microgravity technologies, whose fundamentals were recently developed in experiments [7,12] (see, also, references therein). To explain the experimental vibro-phenomena, the authors extensively employ theoretical concept of vibroequilibria, which were first considered and analysed in the applied mathematical works [1, 2, 8]. The vibroeqilibria (time-averaged, mean liquid shapes in vibrating containers) may dramatically differ from those caused by Newtonian gravitation and surface tension. The difference is clarified by vibrational forces introduced by Blekhman [3]. The extra (in addition to gravitation and surface tension) forces affect both the mean liquid shape and its hydrodynamic stability, i.e., the high-frequency tank vibrations may make the mean free surface unstable, or, contrary, stabilise it. Using the mathematical theory from [1, 2, 8], even though it was based on a rather simple hydrodynamic model of ideal compressible fluids, demonstrates a rather adequate prediction of the experimentally-observed vibrational phenomena.

Along with technologies of acoustical levitation and vibrational control of a limited liquid volume in a shaken tank, there exists another class of contactless (acoustic) techniques in microgravity, whose idea comes from famous experimental observations by Wesseln [16]. These experiments showed that generating an acoustic field in the ullage gas (vapour) makes it possible to destabilise

Key words. Vibroequilibria; variational formalism; interfacial flows.

(stabilise) the liquid-gas interface for certain input acoustic frequencies. For cryogenic two-layer fluids, the destabilisation leads to extensive evaporations of the condensed component, an increase of the mean pressure in the gas domain, and, thereby, it causes the so-called acoustic pumping. A physical theory of the acoustic pumping can be found in [10, 13]. By utilising [5], the present paper develops elements of a mathematical theory of the acoustic destabilisation (stabilisation).

After formulating the non-dimensional mathematical statement of the considered hydrodynamic problem in § 2, which adopts the model of ideal compressible barotropic two-layer fluids, we introduce small parameters (and relations between them) in § 3 to apply the two-timing (separation of slow and fast time) asymptotic technique and derive the free-surface problem describing slow (modulated) motions of the liquid exposed to acoustic loads from the gas side. Mathematically, the latter problem looks identical to those appearing in the liquid sloshing dynamics for a motionless container when Newtonian gravitation, surface tension and acoustic radiation pressure become comparable on the introduced asymptotic scale. This makes it possible to generalise classical results [11] on sloshing of a capillary liquid. § 4 introduces acoustic equilibria (generalisation of capillary equilibria) and spectral theory of linear relative (natural) harmonic standing waves (natural sloshing modes and frequencies). Spectral criterion of stability for the acoustic equilibria is formulated and applied to show that acoustic field can destabilise the flat liquid-gas interface (if exists) for certain input acoustic frequencies. In § 5, we derive an analogy of (pseudo-)potential energy for the acoustic equilibria.

2. Statement of the problem

Following [1], we consider the rigid container

$$Q = Q_1(t) \cap Q_2(t) = \{(x, y, z) | W(x, y, z) < 0\},$$

which is filled by a two-layer fluid where the upper fluid is associated with the ullage (ideal compressible barotropic) gas (domain $Q_1(t)$) but the lower one is an ideal compressible barotropic liquid (domain $Q_2(t)$). The gas and liquid domains are time-dependent and the interface

$$\Sigma(t) = \partial Q_2(t) \cap \partial Q_1(t) = \{(x, y, z) | \xi(z, y, z, t) = 0\}$$

is implicitly specified by the preliminary unknown function ξ such that $\nabla\xi/|\nabla\xi|$ is the outer normal to $Q_2(t)$ on $\Sigma(t)$. The gravitational acceleration is directed downward, against the Ox-axis. Furthermore, we assume an acoustic field generated in $Q_1(t)$ by means of a vibrator on a piece of the time-independent gas boundary

$$S_0 \subset \partial Q_1(t), \quad S_0 \cap \Sigma(t) = \emptyset,$$

which is, in fact, a part of the tank wall contacting with $Q_1(t)$.

As in [1,5], the two-layer fluid dynamics is described by the velocity potentials $\varphi_i(x, y, z, t)$, the pressure $p_i(x, y, z, t)$ and density $\rho_i(x, y, z, t)$ fields in ullage gas ($i = 1$) and liquid ($i = 2$), respectively. Henceforth, the corresponding boundary value problem is considered in the non-dimensional statement, which appears after choosing the characteristic dimension (length) l and time $2\pi/\sigma$, where σ is the circular frequency of the acoustic field in the gas. This non-dimensional mathematical statement takes then the form [5]:

$$\rho_i \nabla\left(\dot{\varphi}_i + \frac{1}{2}(\nabla\varphi_i)^2 + \sigma_*^{-2}\mathrm{Bo}\; x\right) = -\nabla p_i; \quad \rho_i = \left(\frac{p_i}{p_{0i}}\right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t), \tag{1}$$

$$\dot{\rho}_i + \mathrm{div}(\rho_i \nabla\varphi_i) = 0 \quad \text{in} \quad Q_i(t); \quad \int_{Q_i(t)} \rho_i dQ = m_i, \tag{2}$$

$$\partial_n \varphi_i = 0 \quad \text{on} \quad S_i(t); \qquad \partial_n\varphi_i = -\,\dot{\xi}/|\nabla\xi| \quad \text{on} \quad \Sigma(t), \tag{3}$$

$$-p_2 + \sigma_*^{-2}\,(K_1 + K_2) = -\delta_0\, p_1 \quad \text{on} \quad \Sigma(t), \tag{4}$$

$$-\frac{(\nabla W, \nabla\xi)}{|\nabla W||\nabla\xi|} = \cos\alpha \quad \text{on} \quad \partial\Sigma(t), \tag{5}$$

$$\rho_1\; \partial_n\varphi_1 = \varepsilon\,\mu_0\, k^{-1}\, V(x,y,z)\, \sin t \quad \text{on} \quad S_0, \tag{6}$$

where $S_i(t) = \partial Q \cap \partial Q_i\,(i = 1, 2)$ are the time-depending wetted (contacted) walls of Q by gas and liquid, respectively, $\partial\Sigma(t)$ is the contact (gas-liquid-tank) line (curve), α is the contact angle (we assume that α =const), K_i are the main curvatures of $\Sigma(t)$, ρ_{0i} are the mean densities of gas and liquid, respectively, γ_i are the adiabatic indices for the barotropic fluids, p_{0i} are the non-dimensional mean (static) pressures in the fluids ($i = 1, 2$), m_1 and m_2 are (constant) masses of gas and liquid, respectively; the dot implies the time-derivative and ∂_n is the (outer) normal derivative. Furthermore, $\sigma_* = \sigma\sqrt{\rho_{02}l/T_s}$ is the non-dimensional (normalised) acoustic frequency, where T_s is the surface tension, $\mathrm{Bo} = gl^2\rho_{02}/T_s$ is the Bond number, where $\boldsymbol{g}$ is the gravity acceleration, $k = \sigma l/c$ is the wave number of the acoustic field in the gas, where c is the sound speed in the gas, $\delta_0 = \rho_{01}/\rho_{02} \ll 1$ is the ratio between the mean densities.

Originally, $V_0(x,y,z)\sin(\sigma t)$ is the given dimensional distribution of the normal velocity on the acoustic vibrator $S_0 \subset S_1$ but the normalisation introduces the non-dimensional distribution $V = V_0/\mathrm{sup}|V_0|$, the small parameter $\varepsilon = \sup|V_0|/(c\mu_0) \ll 1$ (ratio of the maximum vibration velocity and the sound speed, an analogy of the Mach number) as well as the non-dimensional parameter $\mu_0 = O(1)$.

Remark 2. *Since the fluids (gas and liquid) are barotropic, equations (1) admit the Lagrange-Cauchy integral. However, this does not simplify the asymptotic procedure below.*

3. Asymptotic almost-periodic solution of (1)-(6)

The problem (1)-(6) contains two small parameters, one of which is associated with the density ratio $\delta_0 \ll 1$ but the second small parameter is the non-dimensional value $\sigma_*^{-2} \ll 1$, which physically implies that the sound frequency is much larger than the lowest eigenfrequency of the interfacial (sloshing) waves [11]. To construct an almost-periodic solution, we assume the following asymptotic relations between the two small parameters

$$\rho_{01}/\rho_{02} = \delta_0 = \mu_1\varepsilon, \quad \mu_1 = O(1); \qquad \sigma_*^{-2} = \mu\;\mu_1\varepsilon^3, \quad \mu = O(1). \tag{7}$$

The asymptotic procedure adopts the multi-timing technique of vibrational mechanics [3], which introduces fast and slow time scales such that the fast time is associated with the dimensionless time t appearing in the inhomogeneous condition (6) (expresses the input acoustic signal) and the slow time scale τ should be proportional to the square-root of the dimensionless forces of potential type (Newtonian gravitation and surface tension). The latter forces are of the order $O(\varepsilon^3)$; they appear in the dynamic interface condition (4) and the Euler equations (1). Therefore, the slow time is defined as $\tau = \varepsilon^{3/2}t$ and the non-dimensionless solution of (1)-(6), (7) can be posed in the following form

$$\begin{aligned} \varphi_i &= \sum_{k=0}^{\infty} \varepsilon^{k/3}\ \varphi_i^{(k)}(x,y,z,t,\tau), \quad p_i = \sum_{k=0}^{\infty} \varepsilon^{k/3}\ p_i^{(k)}(x,y,z,t,\tau), \\ \rho_i &= \sum_{k=0}^{\infty} \varepsilon^{(k/3)}\ \rho_i^{(k)}(x,y,z,t,\tau), \quad \xi = \sum_{k=0}^{\infty} \varepsilon^{k/3}\ \xi_k(x,y,z,t,\tau). \end{aligned} \tag{8}$$

Substituting (8) into (1)-(6) and using the standard multi-timing technique, which separates t and τ, derives the free-surface (sloshing-type) problem

$$\Delta\varphi = 0 \quad \text{in } \langle Q_2\rangle(\tau), \tag{9}$$

$$\partial_n\varphi = 0 \quad \text{on } \langle S_2\rangle(\tau), \tag{10}$$

$$\partial_n\varphi = -\partial_\tau\zeta/|\nabla\zeta| \quad \text{on } \langle\Sigma\rangle(\tau), \tag{11}$$

$$\begin{aligned} &\partial_\tau\varphi + \frac{1}{2}(\nabla\varphi)^2 + \mu\mu_1\,(\text{Bo}\ x - (K_1+K_2)) + \\ &\quad + \frac{\mu_1}{4}\left(k^2\Phi^2 - (\nabla\Phi)^2\right) = \text{const} \ \ \text{on } \langle\Sigma\rangle(\tau), \\ &\quad - \frac{(\nabla W, \nabla\zeta)}{|\nabla W||\nabla\zeta|} = \cos\alpha \ \ \text{on } \partial\langle\Sigma\rangle(\tau); \quad \int_{\langle Q_2\rangle} dQ = \text{const} \end{aligned} \tag{12}$$

subject to

$$\begin{aligned} &\Delta\Phi + k^2\Phi = 0 \ \ \text{in } \langle Q_1\rangle(\tau); \\ &\partial_n\Phi = 0 \ \ \text{on } \langle S_1\rangle(\tau) \cup \langle\Sigma\rangle(\tau); \\ &\partial_n\Phi = \mu_0\frac{V(x,y,z)}{k} \ \ \text{on } S_0, \end{aligned} \tag{13}$$

which describes the wave function Φ in the slowly changing gas domain $\langle Q_1\rangle(\tau)$.

Here, $\langle\cdot\rangle$ denotes averaging by the fast time t and, therefore, $\langle Q_2\rangle(\tau)$, $\langle S_2\rangle(\tau)$ and $\langle\Sigma\rangle(\tau)$ are the fast-time averaged liquid domain, wetted tank surface and interface, respectively. The boundary value problem (9)-(13) couples the main terms of the asymptotic representation (8)

$$\begin{aligned} &\varphi_2 = \varepsilon\ \varphi(x,y,z,\tau) + o(\varepsilon); \\ &\varphi_1 = \varepsilon^{2/3}\ \Phi(x,y,z,\tau)\sin t + O(\varepsilon); \\ &\xi = \zeta(x,y,z,\tau) + o(\varepsilon), \end{aligned} \tag{14}$$

which are also independent of t.

Remark 3. *The boundary value problem (9)-(12) is of the mathematically identical structure to the classical sloshing problem of a capillary liquid but with extra pseudo-differential terms in the dynamic boundary condition associated with Φ appearing as solution of the Neumann boundary value problem (13). These extra terms can be interpreted as the acoustic radiation pressure. The radiation pressure parametrically depends on the slowly-varying interface $\langle\Sigma\rangle(\tau)$.*

4. Acoustic equilibria and relative harmonic waves

If the -time averaged interface does not depend on the slow time τ, i.e.

$$\langle\Sigma\rangle = \Sigma_0 : \zeta_0 = \zeta_0(x,y,z) = 0, \langle Q_i\rangle = \langle Q_i\rangle_0 \ (i=1,2),$$
$$\varphi = 0, \quad \Phi = \Phi_0(x,y,z),$$

the problem (9)-(13) reduces to the stationary boundary problem

$$\begin{aligned} &-\mu(K_1+K_2) - \mu \text{ Bo } x + \frac{1}{4}\left(k^2\ \Phi_0^2\ - (\nabla\Phi_0)^2\ \right) = \text{const} \ \text{ on } \Sigma_0, \\ &-\frac{(\nabla W, \nabla\zeta_0)}{|\nabla W||\nabla\zeta_0|} = \cos\alpha \ \text{ on } \partial\Sigma_0; \quad \int_{\langle Q_2\rangle_0} dQ = \text{const}, \end{aligned} \tag{15}$$

where Φ_0 comes from the Newman boundary value problem

$$\begin{aligned} &\Delta\Phi_0 + k^2\ \Phi_0 = 0 \ \text{ in } \langle Q_1\rangle_0; \\ &\partial_n\Phi_0 = 0 \ \text{ on } \langle S_1\rangle_0 \cup \Sigma_0; \\ &\partial_n\Phi_0 = \mu_0\frac{V(x,y,z)}{k} \text{ on } S_0, \end{aligned} \tag{16}$$

$(S_0 \cup \Sigma_0 \cup \langle S_1\rangle_0 = \partial\langle Q_1\rangle_0)$. Equality (15) expresses a balance between surface tension, gravitation and the Langevin acoustic radiation. Following [5], solution of (15), (16) (surface Σ_0 and wave function Φ_0) is called the *acoustic equilibrium.*

Remark 4. *For the introduced asymptotic relations (7), the time-averaged (mean) surface Σ_0 may dramatical differ from the capillary surface. The Langevin acoustic radiation can also influence stability of Σ_0 as well as the natural sloshing frequencies and modes by (9)-(13), which are, in fact, small harmonic waves relative to Σ_0.*

Suppose Σ_0 admits the singe-valued representation, $x = H_0(y,z)$, and linearise (9)–(13) relative to the acoustic equilibrium Σ_0. Furthermore, we consider the natural sloshing modes (H,ψ,Ψ) and frequencies (ω), which correspond to the harmonic solution

$$h = \exp(i\omega\tau)H(y,z); \quad \varphi = i\omega\exp(i\omega\tau)\psi(x,y,z), \quad \Phi = i\omega\exp(i\omega\tau)\Psi(x,y,z)$$

of the linearised problem. The result is the spectral boundary problem with respect to H and ψ

$$\Delta\psi = 0 \text{ in } \langle Q_2\rangle_0; \ \partial_n\psi = 0 \text{ on } \langle S_2\rangle_0; \ \partial_n\psi = \frac{H}{(1+(\nabla H_0)^2\ )^{1/2}} \text{ on } \ \Sigma_0, \tag{17}$$

$$-\omega^2\ \psi + \mu_1\mu A H = 0 \ \text{ on } \ \ \Sigma_0, \tag{18}$$

where ω^2 is the spectral parameter and the linear operator $A = A_1 + A_2$ takes the form

$$
\begin{aligned}
AH &= [A_1 H] + [A_2 H] = \\
&= \left[-\mathrm{div} \left\{ \frac{\nabla H}{(1 + (\nabla H_0)^2\ )^{1/2}} - \frac{(\nabla H, \nabla H_0)\nabla H_0}{(1 + (\nabla H_0)^2\ )^{3/2}} \right\} \right] + \\
&+ \left[\frac{1}{2\mu} \left\{ k^2\ \Phi_0\ \Phi_{0x}\ H - (\nabla\Phi_0, \nabla\Phi_{0x})\, H + \right.\right. \\
&\left.\left. + k^2\ \Phi_0\ \Psi - (\nabla\Phi_0, \nabla\Psi) \right\} + \mathrm{Bo}\, H \right],
\end{aligned}
\tag{19}
$$

$$
\begin{gathered}
\frac{W_y H_y + W_z H_z}{|\nabla_2 W|} = \frac{W_y H_{0y} + W_z H_{0z}}{|\nabla_2 W|} \frac{(\nabla H, \nabla H_0)}{(1 + (\nabla H_0)^2\ )^{1/2}} \ \text{on } \partial\Sigma_0; \\
\int_{\Sigma_0} H dy dz = 0,
\end{gathered}
\tag{20}
$$

$$
\begin{aligned}
&\Delta\Psi + k^2\Psi = 0 \ \text{ in } Q_0; \quad \partial_n\Psi = 0 \ \text{ on } \langle S_1\rangle_0 \cup S_0, \\
&\partial_n\Psi = \frac{\Phi_{0xx}H - \Phi_{0z}H_z - \Phi_{0y}H_y - [\Phi_{0xy}H_{0y} + \Phi_{0xz}H_{0z}]H}{(1 + (\nabla H_0)^2\ )^{1/2}} \ \text{ on } \Sigma_0.
\end{aligned}
\tag{21}
$$

One can study spectral properties of the pseudo-differential operator A and show that it is self-conjugated and has a real pointer spectrum with only a finite set of negative eigenvalues. The following theorem establishes main properties of (17), (18) with the operator (19)–(21).

Theorem 1. *Let H_0, Φ_0 be a solution of the acoustic equilibria problem (9)-(13) such that $H_0 \in C^2(p\Sigma_0)$ and $\Phi_0 \in C^2(\langle Q_1\rangle_0 \cup \Sigma_0)$ (here, $p\Sigma_0$ is the projection of Σ_0 on the Oyz plane). Then*

1. *The spectral boundary problem (17)-(21) has a real pointer spectrum consisting of eigenvalues and $\{H_n\}$ is the functional basis in the factor-space $L_2(p\Sigma_0)/const$.*
2. *The set of negative eigevalues $\{n|\, \omega_n^2 < 0\}$ is finite.*

Proof. Introduce the auxiliary Steklov-Poincáre operator $T : H \to \psi|_{\Sigma_0}$, which is defined by the Neumann problem (17). This operator T is precompact and invertible on the dense set in the factor-space $L_2(p\Sigma_0)/\mathrm{const}$. The boundary condition (18) yields the spectral equation

$$
C_0(\omega^2)H = (\mu\mu_1 A - \omega^2 T)H = 0. \tag{22}
$$

Spectrum of (22) coincides with spectrum of the original problem (17)-(21).

Consider operator A_1, defined by formulas (19). It appears when analysing the eigenoscillations of the capillary liquid and is unbounded, self-conjugate and positive in $L_2(p\Sigma_0)/\mathrm{const}$. Let us introduce the auxiliary operators C_1 and C_2 as

$$
C_1(\omega^2) = \omega^2 A_1^{-1} T - \mu\mu_1(E + A_1^{-1}A_2) = C_2(\omega^2) - \mu\mu_1 E,
$$

where C_1 is due to the action of A_1^{-1} from the left on operator C_0 in (22). The operator $C_2(\omega^2)$ is precompact in the factor-space $L_2(p\Sigma_0)/\mathrm{const}$. If ω^2 is the eigenvalue of (22), then $\mu\mu_1$ is the eigenvalue of the self-conjugate operator C_2, and, therefore, ω^2 is the eigenvalue of the original spectral problem (17)-(21). Because T and A are self-conjugate operators, their eigenvalues are real.

Regular set of the spectral boundary problem (17)-(21) is not empty and contains, at least, complex numbers with non-zero imaginary components. For a regular point ω_0^2, equation (22) is equivalent to the spectral equation

$$\left(C + (\omega^2 - \omega_0^2)^{-1} E\right) H = 0$$

where $C(\omega_0^2) = C_1(\omega_0^2)^{-1} A_1^{-1} T$ is the compact operator in $L_2(p\Sigma_0)$. Because C is compact, its pointer spectrum consists of eigenvalues. As a consequence, the first assertion of the theorem holds true.

All eigenvalues of $A_1^{-1}T$ are positive and follow from the spectral boundary problem on the natural sloshing modes and frequencies of the capillary liquid, i.e. for all admissible H, the inequality

$$(A_1^{-1} T H, H) > 0$$

holds true. Therefore,

$$\omega_n^2 = \mu\mu_1((H_n, H_n) + (A_1^{-1} A_2 H_n, H_n))/(A_1^{-1} T H_n, H_n),$$

where $(H_n, H_n) = 1$, $(A_1^{-1} T H_n, H_n) > 0$. Because $A_1^{-1} A_2$ is compact and $\{H_n\}$ is the functional base in $L_2(p\Sigma_0)$, then $(A_1^{-1} A_2 H_n, H_n) \to 0, n \to \infty$. Therefore, the second assertion holds.

Corollary 4.2 a. The acoustic equilibria may blow up only due to a finite set of linearly-independent perturbations.
Corollary 4.2 b. The acoustic equilibria are stable, if and only if, all eigenvalues $\{\omega_n^2\}$ of A are positive.

The second corollary is the same as the so-called spectral stability criteria, which was already used in [11] for analysing the stability of the capillary equilibria. The stability was investigated by studying the spectrum of the A_1-type operator.

Example 1. (The flat acoustic equilibrium.) The flat capillary surface in an upright cylindrical tank is realised for the contact angle $\alpha = \pi/2$. The flat Σ_0 is also possible for the acoustic equilibria when acoustic vibrator on S_0 generates a planar standing wave, namely, when

$$V_0(x, y, z) = V_0 = \text{const} \quad \left(\varepsilon = -\frac{V_0}{c \ \ \sin(kh_1)}, \mu_0 = -\sin(kh_1), V(y, z) = 1\right).$$

The acoustic equilibrium is then associated with the following solution

$$H_0(y, z) \equiv 0; \quad \Phi_0(x, y, z) = k^{-2} \cos(kx). \tag{23}$$

According to [11, 14], the flat capillary surface corresponds to a unique solution of the capillary problem in an upright circular cylinder, if and only if, $\text{Bo} > \kappa_{11}^2$, where κ_{11} is the minimum root of $J_1'(\kappa_{11}) = 0$ ($J_p(\cdot)$ is the Bessel function of the first kind). Let us pose solutions of the nonlinear boundary value problem (15), (16) as the Fourier series by

$$h_{pq}(r, \theta) = J_p(\kappa_{pq} r) {}^{\sin}_{\cos}(p\theta)$$

in the cylindrical coordinate system, i.e.

$$H_0(r,\theta) = \sum_{pq\neq 00} \eta_{pq}\ h_{pq}(r,\theta), \tag{24}$$

and

$$\Phi_0(x,y,z) = k^{-2}\cos(kx) + \sum_{pq\neq 00} \chi_{pq}\ b_{pq}(x)\ h_{pq}(r,\theta) + \chi_{00}\ \cos(k(x-h_1)), \tag{25}$$

where

$$b_{pq}(x) = \begin{cases} -\dfrac{\cosh(\phi(x-h_1))}{\cosh(\phi h_1)\ \phi\ \tanh(\phi h_1)}, & \kappa_{pq} > k, \\ -\dfrac{\cos(\phi(x-h_1))}{\cos(\phi h_1)\ \phi\ \tan(\phi h_1)}, & \kappa_{pq} < k, \end{cases} \qquad \phi = \sqrt{|\kappa_{pq}^2 - k^2|},$$

in which η_{pq}, χ_{pq} are the unknown coefficients.

Each index pq corresponds to two unknown coefficients for asymmetric solutions and one for symmetric ones $h_{pq}(r,\theta)$, namely,

$$\eta_{pq}\ h_{pq}(r,\theta) = \begin{cases} \eta'_{pq}\ J_p(\kappa_{pq} r)\sin p\theta + \eta''_{pq}\ J_p(\kappa_{pq} r)\cos p\theta, & p\neq 0, \\ \eta_{0q}\ J_0(\kappa_{0q}), & p = 0. \end{cases} \tag{26}$$

Inserting (24) and (25) into equations (15) and (16) and using the Fredholm alternative leads to an infinite system of nonlinear equations with respect to $\eta = \{\eta_{pq}\}$. To within the $o(||\eta||)$-quantities, we have the equalities

$$G_{\alpha\beta} = C_{\alpha\beta}\ \eta_{\alpha\beta} + o(||\eta||) = 0, \tag{27}$$

where

$$C_{pq} = \mu(\mathrm{Bo}\ + \kappa_{pq}\ ) + \frac{1}{2}\ b_{pq}(0), \quad p = 0,1,...;\ q = 1,2,... \tag{28}$$

(C_{pq} are the eigenvalues of the operator A).

The system (27) admits the trivial solution $\eta = 0$, which corresponds to the flat acoustic equilibrium. Trivial solution is stable as $C_{pq} > 0$. When there is an index pq, such that $C_{pq}(k) = 0$, the trivial solution may not become unique. For the eigenvalues with $p \neq 0$, two equations in (27) do not have linear components at η_{pq} but the eigenvalues $C_{0q},\ q = 1,2,\ldots$ have the single multiplicity. In the latter case, the Krasnoselsky theorem [9] gives the sufficient condition of bifurcation of the trivial solution.

5. Pseudo-potential energy of acoustic equilibria

The above example shows that finding the stable acoustic equilibria from its differential statement (15), (16) can be efficient when interface Σ_0 coincides with the capillary surface. If the acoustic equilibrium Σ_0 differs from the capillary surface, identifying solutions of (15), (16) and studying their stability may become a rather complicated task. For the capillary surface, this task sufficiently simplifies by employing the potential energy functional whose minima correspond to the stable liquid shapes. Finding these shapes reduces to a direct numerical minimisation of the potential energy functional.

Theorem 1 in [1] states that the smooth solution of (1)-(6) can follow from necessary extrema condition of the functional

$$G(\xi,\varphi_i,\rho_i)=\int_{t_1}^{t_2}\left\{\int_{Q_2}\rho_2\left(\frac{(\nabla\varphi_2)^2}{2}-U_2(\rho_2)-\mu\mu_1\,\varepsilon^3\ \mathrm{Bo}\ x\right)dQ-\right.$$
$$-\,\mu\mu_1\,\varepsilon^3\ \left(|\Sigma|-\cos\alpha|S_2|\right)+ \qquad (29)$$
$$\left.+\,\varepsilon\int_{Q_1}\rho_1\left(\frac{(\nabla\varphi_1)^2}{2}-U_1(\rho_1)-\mu\mu_1\,\varepsilon^3\ \mathrm{Bo}\ x\right)dQ\right\}dt$$

subject to (1)-(3), (6) and for small variations

$$\delta\xi|_{t_1,t_2}=0,\qquad \delta\rho_i|_{t_1,t_2}=0 \qquad (30)$$

where $p_i\ =\rho_i^2\ dU_i\ /d\rho_i$.

By using the multi-timing technique, one can show that

$$\langle G(\xi,\varphi_i,\rho_i)\rangle=\mathrm{const}+\varepsilon\,\mathcal{G}(\zeta,\varphi)+\mathrm{O}(\varepsilon^{4/3}),$$

where

$$\mathcal{G}(\zeta,\varphi)=\int_{\tau_1}^{\tau_2}\left\{\int_{\langle Q_2\rangle}\left(\frac{(\nabla\varphi)^2}{2}-\mu\mu_1\mathrm{Bo}\,x\right)dQ-\right.$$
$$-\,\mu\mu_1\left(|\langle\Sigma\rangle|-\cos\alpha|\langle S_2\rangle|\right)+ \qquad (31)$$
$$\left.+\,\frac{\mu_1}{4}\int_{\langle Q_1\rangle}\left(k^2\Phi^2-(\nabla\Phi)^2\right)dQ-\frac{\mu_0\mu_1}{2k}\int_{S_0}\Phi V(x,y,z)dS\right\}d\tau,$$

where

$$\int_{\langle Q_2\rangle}\frac{(\nabla\varphi)^2}{2}dQ$$

implies the pseudo-kinetic energy for the sloshing problem (9)–(13) but the remaining quantities can be interpreted as the minus pseudo-potential energy.

Theorem 2. *The problem on the stable acoustic equilibria* $\Sigma_0\ :\ \ \zeta_0=0$ *is equivalent to identifying the minima of the functional*

$$\Pi(\zeta_0)=\mu\left(|\Sigma_0|+\cos\alpha|\langle S_1\rangle|+\int_{\langle Q_2\rangle_0}Bo\,x dQ\right)+$$
$$+\left(\frac{1}{4}\int_{\langle Q_1\rangle_0}(k^2\Phi_0^2-(\nabla\Phi_0)^2)dQ+\frac{\mu_0}{2k}\int_{S_0}V(x,y,z)\Phi_0 dS\right)= \qquad (32)$$
$$=-\mathcal{G}(\zeta_0(x,y,z),\Phi_0(x,y,z)),$$

where Φ_0 *is the solution of (16) subject to the volume conservation condition*

$$\int_{\langle Q_2\rangle_0}dQ=const.$$

The proof comes from computing the second variation by H_0 of the functional $\Pi(x-H_0)$. The second variation by Σ_0 for the surface tension quantities was already derived in [11] (chapter 1). The first variation by Φ_0 is equal to zero restricted to (16) but the first variation by H_0 leads to equation (15), which links Φ_0 and H_0. Furthermore,

$$\delta^2\Pi = \mu^{-1}\int_{p\Sigma_0}(A\delta H, \delta H)dydz,$$

where A is the operator by (19)-(21). Condition $\delta^2\Pi > 0$ is equivalent to the spectral stability criteria 4.2 a.

6. Conclusions

By applying the fast-time averaging of the non-dimensional free-interface problem for two compressible fluids, the mathematical theory of levitating drops in [5] is generalised to study how acoustic field in the ullage gas may affect the mean (time-averaged) liquid-gas interface (called the acoustic equilibrium) and its stability. The theory includes a spectral theorem on the natural frequencies and modes and a pseudo-potential energy introduced for the acoustic equilibria.

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Bibliography

1. Beyer K. Compressible potential flows with free boundaries. Part I: Vibrocapillary equilibria / K. Beyer, M. Gunther, I. Gawrilyuk, I. Lukovsky, A. Timokha // ZAMM. – 2001. – Vol. 81. – P. 261-271.
2. Beyer K. Variational and finite element analysis of vibroequilibria / K. Beyer, M. Guenther, A. Timokha // Comput. Methods Appl. Math. – 2004. – Vol. 4., No 3. – P. 290-323.
3. Blekhman I.I. Vibrational Mechanics. Nonlinear Dynamic Effects, General Approach, Applications / I.I. Blekhman. – Singapore: World Scientific, 2000.
4. Brandt E.H. Suspended by sound / E.H. Brandt // Nature. – 2001. – Vol. 413. – P. 474-475.
5. Chernova M. Differential and variational formalism for an acoustically levitating drop / M.O. Chernova, I.A. Lukovsky, A.N. Timokha // Journal of Mathematical Sciences. – 2017. – Vol. 220, Issue 3. – P. 359-375.
6. Eberhardt R. Acoustic levitation device for sample pretreatment in microanalysis and trace analysis / R. Eberhardt, B. Neidhart // Fresenius' J. Anal. Chem. – 1999. – Vol. 365. – P. 475–479.
7. Fernandez J. The CFVib experiment: control of fluids in microgravity with vibrations / J. Fernandez, P. Salgado Sanchez, I. Tinao, J. Porter, J.M. Ezquerro // Microgravity Science and Technology. – 2017. – Vol. 29, Issue 5. – P. 351-364.
8. Gavrilyuk I. Two-dimensional variational vibroequilibria and Faraday's drops / I. Gavrilyuk, I. Lukovsky, A. Timokha // ZAMP. – 2004. – Vol. 55. – P. 1015-1033.
9. Krasnoselsky M.A. Topological Methods in the Theory of Nonlinear Integral Equations / M.A. Krasnoselsky. – New-York: Pergamon Press, 1964.
10. Lukovskii I.A. Stabilization of liquid-gas interface in the presence of interaction with acoustic fields in the gas / I.A. Lukovskii, A.N. Timokha // Fluid Dynamics. – 1991. – Vol. 26, Issue 3. – P. 382-388.
11. Myshkis A. Low-gravity Fluid Mechanics: Mathematical Theory of Capillary Phenomena / A. Myshkis, V. Babskii, N. Kopachavskii, L. Slobozhanin, A. Tiuptsov. – Berlin: Springer-Verlag, 1987.
12. Sanchez P.S. Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity / P. Salgado Sanchez, V. Yasnou, Y. Gaponenko, A. Mialdun, J. Porter, V. Shevtsova // J. Fluid Mechanics. – 2019. – Vol. 865. – P. 850-883.

13. Timokha A. Influence of sound on the normal modes of oscillation of a liquid-gas interface in a bounded volume / A. Timokha // Acoustical Physics. – 1993. – Vol. 39. – P. 187-189.
14. Ural'tseva N.N. Solvability of the capillary problem II / N.N. Ural'tseva // Vestn. Leningr. Univ. Math. – 1980. – Vol. 8. – P. 151-158.
15. Weber R.J.K. Acoustic levitation: recent developments and emerging opportunities in biomaterials research / R.J.K. Weber, C.J. Benmore, S.K. Tumber, A.N. Tailor, C.A. Rey, L.S. Taylor, S.R. Byrn // Eur. Biophys. J. – 2012. – Vol. 41, No. 4. – P. 397-403.
16. Wesseln Ph.S Acoustic pumping in cryogenic liquids / Ph.S. Wesseln // Design News. – 1967. – Vol. 22. – P. 96-101.

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